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INVESTMENTS IN PRODUCTIVITY UNDER MONOPOLISTIC COMPETITION: LARGE MARKET ADVANTAGE

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Investments in productivity under Monopolistic Competition: Large market advantage

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We study endogenous productivity in monopolistic competition with general (unspecified) utility/investment functions; a bigger investment yields smaller marginal cost. Then the equilibrium investments increase with the market size if and only if the utility (realistically) generates *increasing demand elasticity* and this increase can be abrupt (threshold effect). However, this technological advantage of a bigger country/city is different from a welfare advantage. To fit social optimality conditions, a governmental taxation/subsidy should be non-linear. Our extensions include comparisons among industries with different characteristics, exogenous technological progress, heterogeneous firms.

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Keywords: investments; monopolistic competition; relative love for variety; country size; heterogeneous firms.

Contents

1 Non-Technical Summary

The cross-countries differences in productivity and quality are noticeable and allow for various explanations. In this theoretical project we study the impact of the market size on investments in productivity and quality. Is it true that in a bigger market firms make bigger investments? How firms' investments correlate with competetiveness, measured by Herfindal-Hirshman Index or by markup? In a heterogenous industry, do more productive firms make bigger investments, or less efficient firms invest more, to compensate initial inﬃciency? First of all, for policy-making our topic may be interesting because of new qualitative understanding of big-country advantages and similar gains from trade. Indeed (unlike Melitz (2003)) we show that gains from trade liberalization consist not only in additional product diversity and "best firms selection" but also in fostering R&D, and thereby productivity and quality. Our classification of markets suggests that there can be industry-specific gains from trade liberalization, that allow to detect industries most favorable for liberalization. Second, some modernizing countries (like Russia) do practice active industrial policy including governmental aid or preferences to some industries: automobiles, agriculture, etc. For such policy it can be interesting, that market outcome in sectors of different kind can include too many or too few firms in the industry, compared to social optimum, in the spirit of Dixit and Stiglitz. If their hypothesis will turn out holding under our more broad setting, then we find additional reasons for industry-specific industrial policy for those industries where the number of firms appears insufficient or excessive. In this case, a theorist should recommend regulations hampering/fostering entry into certain industries satisfying our criterion of "efficient number of competitors" connected with the demand elasticity.

2 Introduction

On ***the empirical*** side of our question, the trade literature reports noticeable cross-countries differences in productivity and related indicators. The stylized facts that we believe to be a challenge for theory are the following: (1) firms operating in bigger markets have lower markups (see e.g. Syverson (2007)); (2) firms tend to be larger in larger markets (see e.g. Campbell and Hopenhayn (2005)); (3) larger economies export higher volumes of each good, export a wider set of goods, and export higher-quality goods (see e.g. Hummels and Klenow (2005)); (4) within an industry there can be considerable firm heterogeneity: firms differ in efficiency, in exporting or not (conditional on high/low efficiency), in wages (see review in Reddings (2011)); (5) firms' investments in productivity are positively correlated with the export status of the firm and its size (see e.g. Aw et al. (2008)). In the IO literature, unlike trade, the focus is not so much on the market size, but rather on "market structure and innovations." Among empirical studies linking competitiveness and innovations, several papers find here a *positive* correlation: Baily and Gersbach (1995), Geroski (1995), Nickell (1996), Blundell at al. (1999), Galdón-Sánchez and Schmitz (2002), Symeonidis (2002).[1] These IO findings about competitiveness are in line with the stylized facts from the trade literature on the market size, at least when we believe that a larger market attracts more firms.

In contrast, on ***theoretical*** side, typical oligopoly settings in IO predict that innovation should *decline* with competition (probably, this discrepancy between theory and evidence stems from the simplifying assumption of *fixed* number of oligopolists in IO, instead of free entry). As to trade theorists, some also predict negative correlation. Here our main question manifests itself as "trade liberalization impact on investments in productivity or quality," in particular, Tanaka (1995) finds from the Salop's circular model that trade liberalization *decreases* the product quality. Somewhat similar is Fan (2005). Different conclusions are in Dasgupta and Stiglitz (1980) that encompass several settings including those where different firms influence each other with their R&D. Within the opposite approach (independent firms), the most important paper driving theory closer to evidence—*positively* related competition and innovations—is Vives (2008), discussed in more detail later on.

To mention other papers with such positive relation, Bustos (2011) studies influence of trade on technology adoption in the classical Melitz model supplemented by endogenous choice of technology (available technologies are discrete). In this case trade liberalization increases the share of firms using a high-investment technology. In Yeaple (2005) the set of technologies chosen is also discrete, every producer chooses also the quality of labour used (skilled or unskilled). Then exporting firms are larger, employ more advanced technologies, pay higher wages, are more productive, and a reduction in trade costs can induce firms to switch technologies.

What seems a restrictive assumption in these studies, is the utility function of very specific form. We prefer to follow Vives (2008), who considers general, i.e., unspecified preferences or demand functions. He explores partial equilibrium with endogenous technology choice: innovative investments or R&D. One of his models is the oligopoly of differentiated goods with free entry, posed in terms of arbitrary demand functions. Vives studies the comparative statics with respect to the market size (and also to product substitutability and entry cost). Under some "regularity condition" for profit concavity and certain restriction on the demand shape, he shows that investments in productivity *increase* with the market size (see Proposition 2 in Vives (2008)). However, strategic interactions, typical in oligopoly, make this analysis cumbersome, which prevents formulating clear necessary and sufficient conditions for the positive effect found, and complicates the extensions.

Our goal is to enrich the theoretical explanations of these empirical regularities in R&D—through extending Vives's approach to the monopolistic competition framework. The first reason for monopolistic competition is future extension to trade, and its immediate trade interpretations. The second reason is to get rid of strategic interactions (unimportant when firms are numerous). Then the model becomes more

[1] On the other hand, Aghion et al. (2005) demonstrate the possibility of *non-monotone* (hump-shape) correlation between competitiveness and innovations.

tractable and we can show more effects than Vives, including even necessary and sufficient conditions for certain market outcomes—in terms of some tractable conditions on utility functions. Notably, we compare equilibria with social optimum, consider regulation, and consider an extension to heterogeneous firms.

Our setting extends Dixit-Stiglitz-Krugman (Dixit and Stiglitz (1977), Krugman (1979)) monopolistic competition with unspecified utility developed as in Zhelobodko et al. (2012)—towards variable technologies as in Vives (2008). We argue that unspecified preferences look more robust than any specific form, especially CES utility, which has beenbroadly criticized (see e.g. Ottaviano et al. (2002), Behrens and Murata (2007)).[2] Importantly, CES specification makes prices and outputs independent of the market size, that prevents a meaningful modelling of endogenous technology. By contrast, our choice of a general equilibrium model, instead of partial equilibrium, is not crucial for our results. They generally remain the same in a quasilinear setting also, as we mention among the extensions.[3]

Specifically, we consider one-sector one-factor monopolistic competition in a closed economy, with endogenous choice of technology in the diversified sector. R&D or other technological choice means that higher investment (fixed cost) entails lower marginal cost, which is presented by a non-specified decreasing "investment function" called also "innovation function". The degree of its convexity becomes a criterion for some outcomes. Similarly, on the consumer side what matters for outcomes is the degree of the concavity of the non-specified elementary utility function. Namely, we use the Arrow-Pratt measure of concavity, known as relative risk aversion, called "relative love for variety" (RLV) in our context, and measuring also the elasticity of the inverse demand. Under sufficiently flat demands, called "sub-convex" in Mrázová and Neary (2012) both (absolute value of) demand elasticity and elasticity of the inverse demand are increasing. We call this case IED or "increasingly-elastic" demands. Instead, for "super-convex" demands both elasticities decrease (DED case), while CES utility provides the borderline case—iso-elastic demand.

Our findings are focused on advantages or disadvantages of large markets for R&D investments. We first study the impact of the market size on equilibrium prices, outputs, mass of firms, and most importantly—on investment in productivity. Rather comprehensive comparative statics is summarized in Table 1 into three main patterns: increasing, constant, or decreasing demand elasticity.

Intuitively, our initial hypothesis was that when a firm sells to a bigger population (like China), it has always *more* incentives to invest in decreasing its variable cost, thus exploiting the economies of scale. However, actual results show that such positive relation holds true only under increasing RLV (i.e., decreasing elasticity of substitution, IED case). By contrast, in CES case RLV remains constant, so investments do not change. In the third case, decreasing elasticity of demand (DED), we observe counter-intuitive negative impact of market size on the investments.

The latter surprising prediction can be explained as follows. Under growing market more competitors enter, then the demand decrease for each variety. In response, under super-convex demand each firm increases its price and *decreases* the output (see Zhelobodko et al. (2012) for similar prediction without investments). Output being always positively related to cost-reducing investment, economies of scale induce paradoxically declining investment—contradicting the aggregate stylized facts. This discrepancy probably means that among industries *super-convex demands are non-typical* (or even absent), and have been considered as not too realistic in Krugman (1979) and other papers.

Other equilibrium characteristics—prices and outputs—follow the same three patterns of Table 1. Similarly to fixed technology case (Zhelobodko et al. (2012)), a bigger market pushes prices down under increasing RLV (IED), makes them increase under DED, and does not affect prices under CES. Outputs always move oppositely to prices, the mass of firms always increases. Here the cost function plays smaller role for classifying the market outcomes, yet, some subcases of our main three patterns relate to markup and size of a purchase. Comparing our findings in Proposition 3 to Zhelobodko et al. (2012), we included now the cases without differentiable producers' responses to the market size, which is quite realistic under

[2]"As a theorist, I'm not used to relying on particular functional forms for results. These are usually called 'examples', not 'theorems'." (Berliant (2006), page 108).

[3]General equilibrium like monopolistic competition appears more suitable for possible extensions to international trade, because in trade income effects do matter.

threshold effects in R&D investments; the responses are even discontinuous. This analysis requires the Milgrom's "ordinal" technique of comparative statics instead of usual differentiation.

Concerning social (non-)optimality of equilibria, under our variable technology we find a similar optimality criterion as in Dixit and Stiglitz (1977) (and as in Dhingra and Morrow (2011) who study optimality in a heterogeneous industry). Namely, the only case of welfare-maximizing equilibrium is iso-elastic utility, this feature holding globally only for CES. By contrast, increasing elasticity of utility (IEU case) induces insufficient competition, measured by the mass of firms, and too big investment. Finally, decreasing elasticity (DEU case) brings the opposite effect: excessive competition and insufficient investment. In particular, rather natural utility functions $u(x) = (x + a)^{\rho} - a^{\rho} - bx$ show IED+DEU properties, and we extend this property to all sums of power functions with natural coefficients.

Such "natural" IED+DEU utilities always bring *insufficient* outputs, similarly to constant-technology finding in Dixit and Stiglitz (1977). We extend this finding and relate it to *insufficient* R&D investments, as in Dasgupta and Stiglitz (1980). However, this discrepancy is partially cured by the growing market, which enhances investments, driving them closer to social optimum (which also increases, see Proposition 6). Thus, *growing market drives equilibrium and social optimum closer to each other* under those types of demand that seem most plausible.[4] This shows one more benefit of markets integration (see similar convergence in Dhingra and Morrow (2011) under fixed investments).

Our new topics include (i) threshold effect and non-smooth comparative static Proposition 3 (based on Milgrom's approach); (ii) governmental regulation, (iii) multi-sector economy, and (iv) firms' heterogeneity.

(**i**) The threshold effect means that we assume such investment function that allows our firms to "jump" abruptly from non-investing in R&D to a noticeable investment, in response to infinitesimally small increase in the market size, bypassing the stage of infinitesimally small investment. It looks quite realistic and we argue that this should be the case under most realistic innovation functions. However, technically this realism costs something. We are forced to deal with profit functions that are not concave globally (which is actually inevitable for modelling R&D, as we show), multiple equilibria, and comparative statics avoiding differentiability, and even continuity, of equilibria responses to the market size. Luckily, the technique from Milgrom and Roberts (1994) can cope with such situations.

(**ii**) When the market is not big enough for negligible discrepancy between the market equilibrium and social optimum—what should the government do? Somewhat surprisingly for common sense, neither linear taxation nor linear subsidizing revenue can help. The reason is that the equilibrium equation differs from the optimum equation only in term of elasticities. Namely at equilibrium the elasticity of revenue equals the elasticity of some "reduced-form" cost function, whereas at optimum the utility elasticity replaces the revenue elasticity. So, to correct optimum, we should modify the revenue elasticity, which is impossible by a linear transform; subsidizing or taxation should be nonlinear.

Another possible regulation can be on the cost side: subsidizing R&D in one or another form. This topic is studied in our Section 6 together with the comparative statics w.r.t. the exogenous technological progress. Additionally, we compare any two industries where the first have more concave elementary utility, i.e., more love for variety. Naturally, it yields smaller firms and consequently, smaller R&D.

(**iii**) An important extension is multiple industries interacting with each other under endogenous technology. This seems a very hard theoretical issue but actually under additive utilities it is not. We achieve such extension (see similar findings under exogenous technology in Zhelobodko et al. (2010)).

(**iv**) The most important extension is the selection of the best producers by larger markets, that means a heterogeneous model a'la Melitz (2003), but with endogenous technology and variable elasticity of substitution. The main finding, agreeing with the same conclusion in a fixed-technology heterogeneous model Zhelobodko et al. (2012), is that in any IED market the cutoff increases, so the productivity of the whole economy increases due to selection effect. Now the mechanism of such increase works through investments in R&D. Additionally, we see that this selection turns out *enforced* by more active R&D in the strongest firms, who increase their R&D comparable to smaller markets, so stimulation of innovation works

[4] This angle of discussing the big-market benefits looks more appropriate than traditional IO question about growing or decreasing R&D under expanding market.

in the same direction as selection. By contrast, under DED all these effects work oppositely, unfavorably for productivity.

It should be explained that our result on positive effect of the market size on productivity of a heterogeneous industry looks rather similar to that in Melitz and Ottaviano (2008), and really similarly describes the selection effect. However, the important difference is that Melitz-Ottavino model replaces the increasing returns and fixed costs with the choke-price assumption, which *directly* selects the best firms under market expansion under any choke-price utilities. Instead, our approach is more traditional for monopolistic competition and links good or bad market selection to the IED or DED classes of preferences, that may display or not the choke-price property (a finite derivative at zero).

Section 3 presents the model. Section 4 includes comparative statics. Section 5 studies comparison of equilibrium with social optimality, comparative statics of optimum, and regulation. Sections 6 and 7 show the impact of technological progress and discuss the inter-industry comparisons. Section 8 studies multi-sector economy case. Section 9 studies heterogeneity. Section 10 concludes. Most proofs are in Appendices A and B (Sections 11 and 12). Appendix C (Section 13) presents a table of all the notations and their definitions.

3 One-sector economy with endogenous technology

We model now the closed economy with endogenous investments in technology, as in Vives (2008), but with general equilibrium and monopolistic competition. Comparing our setting to the standard Dixit-Stiglitz model (see Dixit and Stiglitz (1977)) with CES, we generalize their approach in two respects: we enable investments in productivity and allow for general (unspecified) elementary utility function.

Our economy exploits one production factor interpreted as labour. There is one sector using this labour (though the same model can also describe a sector within a multi-sector economy, see Zhelobodko et al. (2012)). There are two types of agents: big number L of identical consumers/workers and an endogenous interval $[0, N]$ of identical firms producing varieties of some "differentiated good." Our goal is to show why *different preferences for variety determine different outcomes of competition.*

3.1 Consumer

Each consumer maximizes her utility under the budget constraint through choosing an infinite-dimensional consumption vector (integrable function) $X : [0, N] \to \mathbb{R}_+$, where N is the endogenous mass of firms or the scope (the interval) of varieties. All consumers behave symmetrically, so the consumer's index is redundant.[5] As in Krugman (1979), Vives (1999) and Zhelobodko et al. (2012), the preferences are described by unspecified additive-separable utility function maximized under the budget constraint:[6]

$$\max_X \int_0^N u(x_i)di, \qquad s.t. \quad \int_0^N p_i x_i di \leq w = 1. \tag{1}$$

Here scalar x_i is consumption of i-th variety by any consumer and $X = (x_i)_{i \leq N}$. Further, p_i is the price, $w \equiv 1$ is the normalized wage and profits are neglected, vanishing at the equilibrium.

[5]The assumption on identical consumers seems restrictive, but we argue that it replaces other typical heroic assumptions: either CES utility or quasi-linear utility, both allowing for consumer's aggregation. Could the opposite assumption (heterogeneous consumers) change our main the results? No. Indeed, when a monopolist meets heterogeneous consumers, then, naturally, the markup chosen would be determined by *average* demand and its elasticity increasing or decreasing. So, this generalization amounts to reformulation of comparative statics in terms of the (average) demand characteristic $r(x)$, instead of RLV $r(x)$ defined for utility. Of course, the model should change somewhat, allowing for vector-valued marginal utility of income.

[6]This additively-separable utility class includes CES-function and CARA function from Behrens and Murata (2007) are the special cases.

Another version of our setting is an additive combination of several sectors with sub-utilities u_1, u_2, so that $U = \int_0^{N_1} u_1(x_i)di + \int_0^{N_2} u_2(x_i)di + ...$ In this multi-sector version *all our results remain true, but except for* firms' masses, N_k, that behave somewhat differently. Below we stick to one-sector notation.

Assumption 1 (*suitable* u). *The elementary utility function* $u(\cdot)$ *satisfies*

$$u(0) = 0, \ u'(x_i) > 0, \ u''(x_i) < 0,$$

i.e., it is everywhere increasing, strictly concave. For our results we need not specify restrictions like homothety or CES. Instead, as in Krugman (1979), Vives (1999) and Zhelobodko et al. (2012), our classification of markets uses the Arrow-Pratt measure of concavity defined for any function g *as* $r_g(z) \equiv -\frac{zg''(z)}{g'(z)}$ *and we require*

$$r_u(0) \equiv -\lim_{z \to 0} \frac{z \cdot u''(z)}{u'(z)} \in [0, 1), \quad r_{u'}(z) \equiv -\frac{zu'''(z)}{u''(z)} < 2 \ \forall z \in [0, \infty). \tag{2}$$

Thereby the demand concavity $r_{u'}$ *is restricted, that would guarantee global strict concavity of producer's profit under constant costs, which could ensure equilibria symmetry and uniqueness (see Zhelobodko et al. (2010) and our paragraph about uniqueness in Subsection 3.4). Additionally, we have restricted the inverse-demand elasticity* r_u *at zero, to guarantee equilibria existence. All these natural restrictions are maintained throughout, as well as the following boundary conditions on the "elementary revenue" function defined as* $R_u(z) \equiv zu'(z)$*:*

$$R_u(0) = 0, \ \ \overline{MR} \equiv \lim_{z \to +0} R'_u(z) > 0, \ \ \underline{MR} \equiv \lim_{z \to \infty} R'_u(z) \leq 0. \tag{3}$$

The characteristic r_u of utility is known as "relative risk aversion" in risk theory and is named "relative love for variety" (RLV) in monopolistic competition (Zhelobodko et al. (2012)). Actually, the consumer's program above is formally equivalent to the portfolio program. So, a well-known fact applies: concave u results in balanced mixture of varieties, which becomes symmetric ($x_i = x_j$) under identical prices of varieties. Under symmetry, function $r_u(z) = 1/\sigma(z)$ is the inverse to the elasticity of substitution among varieties and serves as the elasticity of inverse demand for each variety (standardly). This explains its important role for market outcomes. In particular, for CES utility $u = x^\rho$, RLV is constant: $r_u(z) \equiv 1 - \rho$. Therefore under CES (i.e., iso-elastic demand) the equilibrium prices and outputs remain indifferent to the market size, that hides the links which we would like to uncover. In contrast, under increasingly-elastic demand (IED) or decreasingly-elastic demand (DED) we are going to show opposite behavior of investments.

Using our assumptions and the Lagrange multiplier λ, the first order condition (FOC) generates the inverse demand function for $i - th$ variety:

$$p(x_i, \lambda) = \frac{u'(x_i)}{\lambda}. \tag{4}$$

Obviously, p decreases in consumption x_i. Higher marginal utility of income λ also leads to a decrease in demand and thereby the marginal utility of income λ becomes the only market statistic important for producers, a sort of measure of the intensity of competition.

3.2 Producer

On the supply side, we standardly assume that each variety is produced by one firm that produces a single variety. However, unlike classical settings but like in Vives (2008), each producer chooses the technology level. Namely, when spending f units of labour as fixed costs, the total costs of producing q units of output

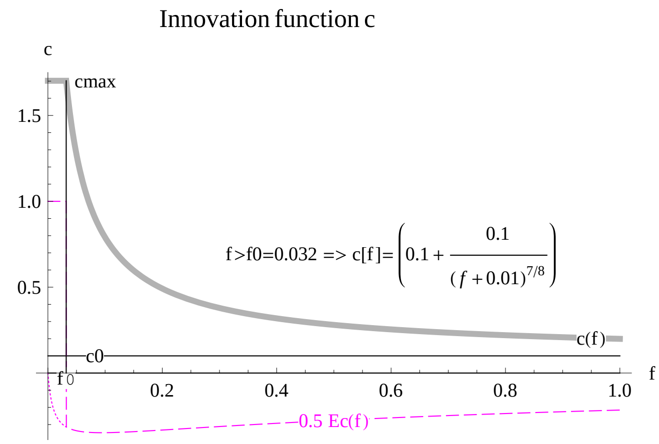

Figure 1: Sample innovation function $c(f) = \min\{1.7, \frac{0.1}{(f+0.01)^{7/8}} + 0.1\}$, $f_0 = 0.032$.

will amount to $c(f)\,q + f$ units of labour. It is natural to assume a non-increasing bounded "innovation function" c, which appears strictly convex where it strictly decreases:[7]

$$c(0) = c_{max}; \quad \lim_{f\to\infty} c(f) \geq c_0 > 0; \quad c'(f) < 0 \Rightarrow c''(f) > 0.$$

It means that: (1) more expensive equipment would incur smaller marginal costs; (2) investment in productivity shows decreasing productivity returns; (3) marginal cost cannot fade to zero: $c \geq c_0 > 0$.[8] Similarly to c_0 assumption, it is reasonable to assume minimal investment requirement $f_0 > 0$ necessary for any business, with or without R&D (this f_0 makes function c truncated at some level c_{max}, see Fig.1). Then R&D is the difference $z = f - f_0$ between total investment f and this technological minimum. Respectively, we call the equilibrium "R&D regime" when this difference z is positive. *We further study mainly the R&D regime*, though possible abrupt switching between the two regimes becomes a new issue, so far contemplated only in Mrázová and Neary (2012), called a "threshold effect." This effect is not due to the kink (point $(f_0, cmax)$ in Fig.1), this kink can be smoothed keeping the threshold. Realistically, a threshold means that either zero or noticeable R&D investment maximizes profit, whereas investing a penny is never profit-maximizing (locally non-concave profit).

Now, using the inverse demand function $p(x_s, \lambda)$ from (4), the profit maximization of $s-th$ producer can be formulated as[9]

$$\pi_s\left(x_s, f_s, \lambda\right) = \left(p(x_s,\lambda) - c(f_s)\right) L x_s - f_s = \left(\frac{u'(x_s)}{\lambda} - c(f_s)\right) L x_s - f_s \to \max_{x_s \geq 0, f_s \geq \hat{f}}.$$

Under continuum of producers, it is standard to prove that each producer s has a negligible effect on the whole market, i.e. the Lagrange multiplier λ can be treated parametrically by each s. This Lagrange multiplier is the natural aggregate market statistic: the bigger is the marginal utility of income λ, the lower is the demand curve and therefore smaller is the profit. Thereby this λ is the "toughness of competition" among the producers of differentiated goods, like a price index in standard Dixit-Stiglitz model (see Zhelobodko et al. (2012)).

[7] We also call $c(.)$ an *investment function*, because its inverse shows how much one should invest to arrive at some marginal cost c.

[8] This assumption is needed for finite maximum in profit maximization.

[9] Standardly, maximization of monopolistic profit w.r.t. price or quantity gives same results.

Profit maximization w.r.t. supply x and investment f yields FOC:

$$\frac{u''(x_s)x_s + u'(x_s)}{\lambda} - c(f_s) = 0, \tag{5}$$

$$c'(f_s)Lx_s + 1 = 0. \tag{6}$$

These equations are valid under second order condition (SOC), that must hold at least locally. Negative semi-definiteness of the Hessian matrix at the R&D regime amounts to two conditions:

$$-\left(u'''(x_s)x_s + 2u''(x_s)\right) > 0 \Leftrightarrow r_{u'}(x_s) < 2, \tag{7}$$

$$-\left(u'''(x_s)x_s + 2u''(x_s)\right) c''(f_s)x_s > \lambda \left(c'(f_s)\right)^2, \tag{8}$$

the latter being stronger than the former at R&D regime ($c' < 0, c'' > 0$), in the opposite regime only the former applies (being assumed throughout).

For each producer the FOC conditions are the same. So, below we focus only on *symmetric* equilibria, denoting $x_s = x,\ f_s = f\ \forall s$. Thereby, we bypass the delicate question of asymmetric equilibria with non-concave profit, where each producer has two or more equivalent local maxima (see Gorn et al. (2012)). Such equilibria are really possible in such models with a (realistic!) threshold. But luckily, only zero-measure set of parameters L brings related argmaxima multiplicity, as we shall explain. We shall see that with or without our minimal f_0 assumption, the profit function is often globally-non-concave in such R&D models. This fact precludes imposing SOC (8) globally for all λ, it is an "impossible assumption".

3.3 Equilibrium

Entry. Standardly, we assume that firms freely enter the market while their profit remains positive, which implies a zero-profit condition

$$\frac{u'(x)}{\lambda} - c(f) = \frac{f}{Lx}. \tag{9}$$

Labour balance. Under symmetric equilibrium ($f_i = f$, $x_i = x$) labour market clearing means

$$\int_0^N \left(c(f_i)x_iL + f_i\right) di = N\left(c(f)xL + f\right) = L. \tag{10}$$

Summarizing, we define *symmetric equilibrium* as a bundle $(x^*, p^*, \lambda^*, f^*, N^*)$ satisfying:

1) utility maximization (4);

2) profit maximization (5)-(6) and (7)-(8);

3) free entry condition (9) and labour balance (10).

It is straightforward to exclude λ, and rewrite the equilibrium equations in terms of the Arrow-Pratt measure of concavity $r_g(x)$:[10]

Proposition 1. *Equilibrium consumption/investment couple (x^*, f^*) in one-sector economy with positive R&D ($f > f_0$) is the solution to the system*

$$1 - r_u(x) = \frac{Lxc(f)}{f + Lxc(f)}, \tag{11}$$

[10]Main assumptions $1 - r_u(x) > 0$, $2 - r_{u'}(x) > 0$ and $(2 - r_{u'}(x))\, r_c(f) > 1$ are used in our proofs only locally. These assumptions are rather standard for any monopolist. The condition $1 - r_u(x) > 0$ must be fulfilled in equilibrium: the monopolist charges the price on the elastic part of the demand curve. Profit concavity condition $2 - r_{u'}(x) > 0$ guarantees that FOC relates to a local maximum, not minimum. Finally, $(2 - r_{u'}(x))\, r_c(f) > 1$ is also the concavity condition, but for endogenous investments in technology.

$$(1 - r_{\ln c}(f) + r_c(f))\,(1 - r_u(x)) = 1, \tag{12}$$

under the conditions

$$r_u\,(x) < 1, \qquad (2 - r_{u'}\,(x))\,r_c\,(f) > 1. \tag{13}$$

Corresponding equilibrium mass of firms N^* is determined by equation

$$N = \frac{L}{c(f)xL + f}\,, \tag{14}$$

price $p^*$is determined by

$$p = \frac{c(f)}{1 - r_u(x)}\,, \tag{15}$$

and markup $\dfrac{p^* - c(f^*)}{p^*}$ is found from

$$\frac{p^* - c(f^*)}{p^*} = r_u(x^*) = \frac{N^* f^*}{L}\,. \tag{16}$$

Under zero R&D ($f = f_0$) the equilibrium is determined by the same equations without (12) and with $f = f_0$.

3.4 Model reformulation: non-linear cost, thresholds and changing regimes[11]

Here we are going to reduce our model with the endogenous choice of technology—to a model with non-linear fixed technology, like in Zhelobodko et al. (2012).

Namely, in Zhelobodko et al. (2012) a firm producing output $q = x \cdot L > 0$ with fixed cost $f = f_0$ faces total non-linear total cost $\mathbf{C}(q) = V(q) + f$. Total cost was supposed *convex* in several propositions but in our paper it will not be the case. In our context investment f becomes an optimization variable and variable cost V becomes a function of q and f. Then, the firm's optimization can be decomposed into stages:

(1) for any given q, we find investment $\check{f}(q)$ that minimizes costs:

$$\check{f}(q) \equiv \arg\min_{f \geq \hat{f}}[f + qc(f)],$$

(2) we maximize profit w.r.t. q using $\check{f}(q)$.

Thereby we can define the total cost depending only on q as

$$\mathbf{C}(q) \equiv V(q, \check{f}(q)) + \check{f}(q) = c\left(\check{f}(q)\right) \cdot q + \check{f}(q).$$

How this function behaves? It is easy to show that optimal investment increases w.r.t. output in R&D regime ($\check{f}' > 0$), which under assumption $c' < 0$ yields decreasing marginal cost:

$$\left(\check{f}(q)\right)' > 0 \quad \forall q:\ \check{f}(q) - f_0 > 0 \Rightarrow \frac{d}{dq}c\left(\check{f}(q)\right) < 0.$$

Further, using the envelope theorem, the derivative of total cost $\mathbf{C}$ is exactly the marginal cost $c(\cdot)$, thereby the second derivative of total cost $\mathbf{C}$ is negative in R&D regime and bounded:

$$\mathbf{C}'(q) = c\left(\check{f}(q)\right), \quad \mathbf{C}''(q) = c'\left(\check{f}(q)\right) \cdot \check{f}'(q) < 0.$$

[11] Mathieu Parenti did participate in converting our problem to non-linear costs and proposed a promising idea to apply similar conversion to any additional tool optimized by a firm together with output: advertising, quality, or anything else. Then the theorems from Zhelobodko et al. (2012) become applicable to all these topics.

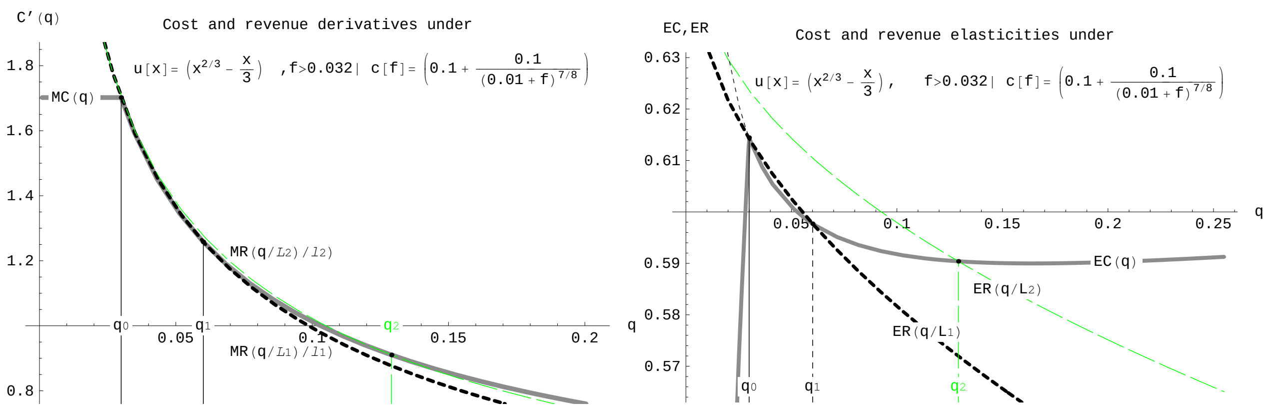

Figure 2: The main equilibrium equation and its prerequisite $MR = MC$ (scaled).

Thus we have arrived at

Remark 1 (Locally concave cost). *Total cost function* $\mathbf{C} = \mathbf{C}(q)$ *(that includes optimal investment) is strictly concave w.r.t. output under R&D regime* ($\check{f}(q) > f_0$).

Concavity is a simple but important result: in contrast with convex costs typical in IO, situations with additional technological parameter—R&D investment—*necessarily generate concave cost* (similar outcome occurs under advertising and other similar investments). This fact is unpleasant, possibly undermining global profit concavity. So, we necessarily step into the shaky grounds of possibly multiple profit argmaxima. Convex cost was typically assumed away in standard oligopoly theory, not only for technical reasons. Indeed, under non-differentiated good, concave cost would necessarily lead to natural monopoly. Even under our differentiated good, such market structure (pure monopoly) *can* be the outcome instead of monopolistic competition, as we argue below.

Main equilibrium equation. Reduction of our investment model to non-linear cost model simplifies the analysis as follows. We divide the producer's FOC by the free-entry condition $R(q/L) = C(q)$ to get rid of λ. Then we arrive at the main equilibrium equation (similar to Zhelobodko et al. (2012) and to our equation (11)) that connects the elasticity of per-purchase revenue $R(x) \equiv x \cdot u'(x)/\lambda$ and the total cost elasticity as

$$\mathcal{E}_R(q/L) \equiv 1 - r_u(q/L) = \mathcal{E}_{\mathbf{C}}(q). \tag{17}$$

Such equilibrium condition is illustrated by the right panel in Fig.2, and one can see the kink marking the switch from non-investing regime to R&D regime. The same kink is present also in the left panel, which shows related Marginal Revenue (MR) $\frac{d}{dq}[q\frac{u'(q/L)}{\lambda}]$ and Marginal Cost (MC) $\mathbf{C}'(q)$ (here the utility is $u = x^{2/3} - x/3$ and the cost function is the same as in Fig.1).

We take the population size $L_1 = 1.49$, then the right panel in Fig.2 shows *two* free-entry equilibria $q_0 \approx 0.030$, $q_1 \approx 0.060$, denoting the output levels where the two curves $\mathcal{E}_R(q/L), \mathcal{E}_{\mathbf{C}}(q)$ intersect. We have adjusted the data so that q_1 coincides with $q_0 \approx 0.030$, the switching point of the regimes, to show that such degenerate outcome may happen at some L_k. More typically, there are three intersections and the middle one means the profit minimum (not an equilibrium). After finding any solution q_k, one can derive related equilibrium competitive pressure $\lambda_k(L_k)$, such that $R'(q_k/L_k)/\lambda_k = \mathbf{C}'(q_k)$ in the left panel.

Further, trying a larger population $L_2 = 2.5 > L_1$, we see that new equilibrium $q_2 \approx 0.129 > q_1$ is bigger, because $\mathcal{E}_R(\cdot)$ is stretched to the right, that makes it higher under the sub-convex demand plotted (unlike DED case). The same transformation makes the marginal-revenue curve R' in the left panel simultaneously wider ($L \uparrow$) and lower ($\lambda \uparrow$). Here the new curve crosses MC at a new point $q_2 > q_1$ (coincidence $q_2 = q_1$ will be shown only under CES preferences). Such increase in L may generally yield switching from non-R&D to R&D regime, which is the case in our picture, if q_0 were an equilibrium. In such cases, the transition of equilibrium involves some *discontinuity* in q; the switching of the regimes is abrupt. We have observed such discontinuity (threshold effect) for many preferences/costs which are not unnatural, not only those in Fig.2.[12]

Equilibrium existence becomes more clear now: an equilibrium q does exist if and only if loci $\mathcal{E}_R(q/L), \mathcal{E}_{\mathbf{C}}(q)$ intersect. In particular, when the demand has both horizontal and vertical intercepts (for instance, $u(x) = (x+a)^\rho - a^\rho - bx$), the revenue elasticity $\mathcal{E}_R(q/L)$ must go from 1 to 0. This guarantees existence *under any costs* because elasticity $\mathcal{E}_{\mathbf{C}}(q)$ anyway goes from zero to some positive value, under assumption $f_0 > 0$. Arguing in this way, we get the following existence theorem.

Remark 2. *When the demand has both vertical intercept $u'(0) < \infty$ and horizontal intercept $\exists x | u'(x) = 0$, an equilibrium exists under any costs. Alternatively, when variable cost $V(q)$ tends to be approximately linear at infinity, an equilibrium exists under any preferences.*

More generally, we will show that an equilibrium exists whenever at the boundaries of $[0, \infty)$ the elasticities of costs and revenue are bounded as

$$\underline{\mathcal{E}_{\mathbf{C}}} \equiv \lim_{q \to 0} \mathcal{E}_{\mathbf{C}}(q) = 0, \quad \overline{\mathcal{E}_{\mathbf{C}}} \equiv \lim_{q \to \infty} \mathcal{E}_{\mathbf{C}}(q) > \overline{\mathcal{E}_R} \equiv \lim_{z \to \infty} \mathcal{E}_R(z). \tag{18}$$

Thus, in these two broad and realistic cases, existence is guaranteed. In the more general case the joint boundary restriction on cost and preferences is less intuitive: it ensures non-positive or decreasing profit when output goes to zero or to infinity (see Zhelobodko et al. (2012) for an alternative conditions of equilibrium existence). Without such a condition, under some reasonable kinds of preferences/costs our monopolistic competition equilibrium really does not exist. Such cases indeed mean impossibility of monopolistic competition. For instance, conventional iso-elastic utility $u(x) = x^\rho$ *cannot* be combined with iso-elastic marginal cost $c(f) = f^{-\alpha}, f > f_0$. Under such combination, each firm tries to increase output to infinity until arriving at an oligopoly (which restricts the demand expectations by strategic interactions) or even monopoly. Thus, the existence restriction on demands/costs is not technical; the named boundary condition really *distinguishes the markets suitable for monopolistic competition* industry structure from others (CES functions being non-suitable). Decrease in marginal cost plays an unimportant role for existence, the positive limit of MC at infinity is important.

Equilibrium uniqueness. By contrast with existence, equilibrium uniqueness is more seriously undermined by costs that are not globally convex. Like in Fig.2, *many realistic preferences generate two or more intersections* of $\mathcal{E}_R(q/L), \mathcal{E}_{\mathbf{C}}(q)$, even under IED assumption (decreasing $\mathcal{E}_R(q/L)$). Of course, only intersections from-above ($\mathcal{E}_R(q/L) > \mathcal{E}_{\mathbf{C}}(q)$) relate to profit maxima, and to become multiple equilibria they must bring the same profit, but it does happen. Moreover, we easily can find parameters u, L generating multiple equilibria for *any* locally super-convex investment function $c(.)$, as we show soon. To exclude equilibria multiplicity, one can try a restriction on the profit-concavity condition like

$$R''(q/L)/\lambda - \mathbf{C}''(q) < 0 \tag{19}$$

but it is not very helpful, because of parameters L, λ. Indeed, using our finding $\mathbf{C}''(q) < 0$, we can prove that *for any demand schedule and any q there exist a couple (L, λ) violating this condition* (this can be understood from the left panel in Fig.2 or just from varying λ). I.e., assumption (19) imposed everywhere would be vacuous. It is better to require similar thing only for equilibrium magnitudes $\lambda(L)$.

More practically, in terms of elasticities the *necessary and sufficient* condition for excluding multiple equilibria is "no-crossing-from-below" at all equilibria:

$$\forall L, q: \ \ \mathcal{E}_R(q/L) = \mathcal{E}_{\mathbf{C}}(q) \Rightarrow \frac{\partial}{\partial q}[\mathcal{E}_R(q/L) - \mathcal{E}_{\mathbf{C}}(q)] < 0,$$

[12]The smaller is the value f_0 truncating the investment function c, the broader is the "jump". In the limiting case $f_0 \approx 0$, this jump is performed between the zero production $q_0 \approx 0$ and some big production q_1, sufficient to justify some investments. In other words, our truncation of $c(\cdot)$ with the help of f_0 is *not the cause* of discontinuous comparative statics and multiple equilibria, it plays just the opposite role.

which is not an intuitive assumption. More understandable condition *sufficient* under increasingly-elastic or sub-convex demands ($\mathcal{E}_R' = -r_u' < 0$) is $\mathcal{E}_{\mathbf{C}}'(q) > 0$. To express this in primitives, we combine the FOC ($\check{f}: \frac{dc}{df} \cdot q = -1$) with the elasticity definition as

$$\mathcal{E}_{\mathbf{C}}(q) = \frac{c\left(\check{f}(q)\right) q}{\check{f}(q) + c\left(\check{f}(q)\right) q} = \frac{1}{1 + \dfrac{\check{f}(q)}{qc\left(\check{f}(q)\right)}} = \frac{1}{1 - \mathcal{E}_c\left(\check{f}(q)\right)}.$$

The left-hand side increases whenever the right-hand side does (recall $\left(\check{f}(q)\right)' > 0$), so, we get

Remark 3. *Under IED, increasingly-elastic investment function* ($c(\cdot): \mathcal{E}_c'(f) > 0 \, \forall f > f_0$) *is sufficient for equilibrium uniqueness and continuity of comparative statics.*

The above statement that comparative statics w.r.t. L shows no discontinuities—is understandable from studying the main equilibrium equation $1 - r_u(q/L) = \mathcal{E}_{\mathbf{C}}(q)$; the unique solution responds continuously to continuous change in L. The restriction $\mathcal{E}_c'(\cdot) > 0$ becomes also necessary, when we want a uniqueness condition working under *all* sub-convex demands (IED) and iso-elastic demands.[13]

In the opposite case, *without uniqueness, there typically arises the threshold effect*: *a big jump of equilibrium in response to infinitesimally small changes in* L. In particular, our (not unnatural) example in Fig.2 shows such jump.

To exclude such jumps, one can look for investment functions increasingly-elastic everywhere. To this end, one *should* use our cost truncation $f_0 > 0$. Otherwise (when we stick to natural restriction $c'(0) < \infty$) the elasticity $\mathcal{E}_c = \frac{fc'(f)}{c(f)} = 0$ at $f = 0$, remaining negative everywhere else, i.e., the elasticity decreases at least at 0.

Thus, it is our truncation of $c(.)$ that helps in some cases to provide increasingly-elastic c and thereby global (for all L) uniqueness of equilibria. Sometimes we shall use this assumption, to avoid formulating multiple asymmetric equilibria arising at the moments of jumps. In other cases, we focus mainly on comparing symmetric equilibria before and after the jump, without the jump itself.[14]

These preliminaries and pictures explain the nature of our main results in the next section. When the equilibrium behaves continuously and smoothly, our results follow from totally differentiating the main equation w.r.t. L. More generally, even under switching R&D/non-R&D regimes, the direction of changes can be found through a technique avoiding differentiability.

4 Equilibrium comparative statics: impact of market size

Our primary goal is to study the impact on productivity induced by the increasing market size measured by population L. This increase can be interpreted as markets integration or population growth, or comparison between cities different in size. Indeed, in proving below certain signs of changes in response to infinitesimally small increase in the market size, we thereby describe comparisons between finitely different cities also (by continuity). In particular, we start now discussing changes in all equilibrium variables: price p, firm size Lx, mass of firms N, investment of each firm f, and total investments in the economy (Nf). Technically, our equilibrium equations determine (x, f, N) as an implicit function of L.

Market size impact under smooth reactions. Before turning to difficult cases, first we consider the case of unique equilibrium and R&D regime at the point studied. We apply total differentiation and rearrangements as in Zhelobodko et al. (2012) to obtain the elasticities of main variables in terms of the concavity of the basic functions and find the elasticities' signs (the proofs are in Appendix B). These signs classify all market outcomes into three patterns and some sub-patterns according to increasing/decreasing concavity measure $r_u(x)$ (increasing elasticity of demand – IED or decreasing elasticity of demand – DED) and to concavity of $\ln c$.

[13] Indeed, without $\mathcal{E}_c'(\cdot) > 0$ one can find an iso-elastic demand $u'(x) = \rho x^{\rho - 1}$, i.e., constant $\mathcal{E}_R \equiv \rho$ which crosses $\mathcal{E}_{\mathbf{C}}$ twice or thrice. Then all IED demands sufficiently close to this $u'(x)$ will also display multiple equilibria.

[14] See Gorn et al. (2012) for comprehensive study of asymmetry without investments.

Proposition 2. *Under unique equilibrium displaying the R&D regime ($f > f_0$), elasticities $\mathcal{E}_*$ of the equilibrium variables w.r.t. market size L in homogenous economy can be expressed in terms of concavity r_* of basic utility and cost functions as:*

$$\mathcal{E}_x = \frac{(1 - r_{\ln c})\,(1 - r_u)}{(2 - r_{u'})\,r_c - 1} \tag{20}$$

$$\mathcal{E}_{Lx} = \frac{r_c r'_u x}{((2 - r_{u'})\,r_c - 1)\,r_u} \tag{21}$$

$$\mathcal{E}_f = \frac{r'_u x}{((2 - r_{u'})\,r_c - 1)\,r_u} \tag{22}$$

$$\mathcal{E}_{Nf} = \frac{(1 - r_{\ln c})^2\,(1 - r_u)^2}{((2 - r_{u'})\,r_c - 1)\,r_c} + \frac{1}{r_c} + r_u \tag{23}$$

$$\mathcal{E}_N = r_u - \frac{(1 - r_{\ln c})\,(1 - r_u)^2}{(2 - r_{u'})\,r_c - 1} = 1 - \frac{r_c r'_u x\,(1 - r_u)}{((2 - r_{u'})\,r_c - 1)\,r_u} \tag{24}$$

$$\mathcal{E}_p = -\frac{r_c r'_u x}{(2 - r_{u'})\,r_c - 1} \tag{25}$$

$$\mathcal{E}_{\frac{p-c}{p}} = \frac{(1 - r_u)\,(1 - r_{\ln c})\,r'_u x}{((2 - r_{u'})\,r_c - 1)\,r_u} \tag{26}$$

and the elasticities' signs/magnitudes can be classified as in Table 1:

Utility patterns:	*DED*		*CED*	*IED*		
L-elasticities of:	$r'_u < 0$		$r'_u = 0$	$r'_u > 0$		
	$r_{\ln c} \leq 1$	$r_{\ln c} > 1$	$r_{\ln c} \neq 1$	$r_{\ln c} > 1$	$r_{\ln c} = 1$	$r_{\ln c} < 1$
purchase size $\mathcal{E}_x$	$\nexists$	< -1	$= -1$	$\in (-1; 0)$	$= 0$	> 0
output $\mathcal{E}_{Lx}$	$\nexists$	< 0	$= 0$	$\in (0; 1)$	$= 1$	> 1
firm's investment $\mathcal{E}_f$	$\nexists$	< 0	$= 0$	> 0	$\in (0; 1)$	> 0
gross investments $\mathcal{E}_{Nf}$	$\nexists$	> 1	$= 1$	$\in (0; 1)$	$= 1$	> 1
mass of firms $\mathcal{E}_N$	$\nexists$	> 1	$= 1$	$\in (0, 1)$	$= r_u \in (0, 1)$	< 1
price $\mathcal{E}_p = -\mathcal{E}_{Lx} \cdot r_u$	$\nexists$	> 0	$= 0$	< 0	$= -r_u \in (-1, 0)$	< 0
markup $\mathcal{E}_{\frac{p-c}{p}} = \mathcal{E}_{\frac{Nf}{L}}$	$\nexists$	> 0	$= 0$	< 0	$= 0$	> 0

In Table 1, $\mathcal{E}_x \equiv \frac{L \cdot dx^*}{x \cdot dL}$ is the elasticity of equilibrium individual consumption of each variety, $\mathcal{E}_f$ - the elasticity of investment per firm, $\mathcal{E}_N$ - the elasticity of mass of firms, $\mathcal{E}_{Lx}$ - the elasticity of total output of one variety, $\mathcal{E}_{Nf}$ - the elasticity of total investment, $\mathcal{E}_p$ - the elasticity of price, $\mathcal{E}_{\frac{p-c}{p}}$ - the elasticity of mark-up.[15]

Discussion. Commenting, we first say that generally these results remind conclusions of Proposition 2 in Vives (2008) obtained for oligopolistic model (related plausible case and important result is colored red in Table 1). However, our table shows in more detail the influence of market size on all variables; Vives mainly focused on investments. Additionally, instead of only the increasing investment case, we find as much as five different patterns of equilibria responses to the market size. For increasing/decreasing investments, the utility characteristic r'_u becomes the determining criterion. Namely, standard CES case

[15]In Table 1 the first column $[r'_u < 0, r_{\ln c} \leq 1]$ was proved to be empty; equilibria here do not exist. We also do not mention the case $r'_u = 0$, $r_{\ln c} = 1$ where equilibria are indeterminate. Existence of equilibria is stated above. Moreover, numerical examples for the middle columns are already constructed.

(iso-elastic demand) is the borderline between markets with increasing (IED) or decreasing (DED) elasticity of demand. Main finding is that the DED class *shows decreasing investments,* whereas under IED, *firms' investments increase in response to growing market.* Additionally, the investment is always positively correlated with output which has clear interpretation: bigger output motivates higher cost-reducing investment. More intriguing is that each firm's larger output is *not* guaranteed for larger market. Why?

The explanation includes prices. They respond to the market size in the same way as under exogenous technology (f, c) studied in Zhelobodko et al. (2010). Our prices also decrease under IED preferences (naturally), but increase under DED preferences. To explain the latter surprising effect, recall what we know from classical model of monopoly: when the demand decreases, a monopolist can charge either lower or higher price, depending upon the demand elasticity. Intuitively, the market adjustment to growing market works like this: in all cases the first step is the increase in the incumbents' profits. Then extra profits invite new firms into the industry and number, N, of varieties increases. This growing competition pushes the marginal utility of income λ up and the inverse demand function is shifted down by the growing denominator λ. This shift, under very convex demand (DED) pushes the price up (unlike IED case), sharply decreasing the quantity x. At the next step, this high price and small quantity of the incumbents invites new firms and pushes the mass of firms N even further upward. This positive feedback makes the elasticity of N bigger than 1. Thus, decrease of both output Lx and investment f happens under DED because the mass of firms grows too fast. It is the excessive competition that makes the output shrink, outweighing the market-size motive to invest in marginal productivity.[16] In contrast, under $r'_u > 0$ (IED) the slow growth of N makes output Lx increasing, that motivates more investment in productivity.

Thus, price and output behavior found in Zhelobodko et al. (2010) generally remains valid under our endogenous technology also, though costs c and f change. Consumption and the mass of firms behave in a slightly new fashion only under $r'_u > 0$, $r_{\ln c} < 1$.

In contrast to utility, the curvature of the *cost* function becomes a criterion only for increasing/decreasing individual consumption of each variety and for markup $M = \frac{p-c}{p}$. Sufficiently big elasticity of cost to investment expressed in condition $r_{\ln c}(f) > 1$ makes the individual consumption decreasing, otherwise it goes up.

Market size impact under non-smooth reactions. Now we turn to study the case of *switching regimes*, that forces us to avoid derivatives (see Fig.2). For this purpose, we apply some lemmas from Appendix A, which are the modified versions of the "new" comparative statics theorems from Milgrom and Roberts (1994), Milgrom and Shanon (1994), Milgrom and Segal (2002). Thus we immediately obtain an analogue of previous Proposition 2, but now without differentiation and even without continuity (enabling jumps).

Proposition 3. *Assume that the market size L increases from some magnitude L_1 to some $L_2 > L_1$ under boundary conditions (18). Then related equilibrium outputs $q_1 = q(L_1)$, $q_2 = q(L_2)$ and investments $f_1 = f(L_1)$, $f_2 = f(L_2)$ exist in both situations. Moreover:*

(i) Under IED (sub-convex demand, i.e., $r'_u(q/L) > 0$ on interval $[q_1/L_1, q_2/L_2]$) the new magnitudes satisfy inequalities

$$q_2 > q_1, \;\; f_2 \geq f_1, \;\; [f_2 > f_0 \Rightarrow f_2 > f_1],$$

i.e., the output increases, the R&D investment $(f - f_0)$ also increases when positive (and anyway non-decreases). Under DED, the impact is the opposite: $q_2 < q_1$, $f_2 \leq f_1$, $[f_1 > f_0 \Rightarrow f_2 < f_1]$.

(ii) The equilibrium prices change in the opposite direction to quantities ($q_i > q_j \Rightarrow p_i < p_j$).

(iii) The equilibrium intensity of competition λ and firms' mass N both increase:

$$\lambda_2 > \lambda_1, \quad N_2 > N_1.$$

[16]Nevertheless, total economy investment Nf always grows because growing mass of firms dominates even when f decreases.

Commenting, we would say that equilibria *discontinuity does not change the direction of changes in output and R&D investments* induced by the market size. Similarly, a stone sliding from a hill can make a jump which does not change the main direction - down. However, economically such jump should look as an abrupt switch from zero R&D to noticeable R&D in response to gradual market changes. Such evolution (or revolution, catastrophe) rather naturally *bypasses the stage of infinitesimally small R&D*. When applied to heterogeneous firms, this idea would mean that there should be a *gap* between a group of firms applying R&D and those who do not apply it.

In our subsequent propositions we shall only refer to R&D regime, without allowing for jumps, which have not been studied sufficiently.

So far, our main conclusion is that under IED (supposed a realistic case by Krugman (1979), and others) a bigger market is favorable for innovations and productivity. But is it thereby favorable for welfare? Not necessarily, as the next section shows.

5 Equilibrium versus social optimum, regulation

Relation between market equilibrium and social optimality is a traditional question since Dixit and Stiglitz (1977); the mass of firms under rather realistic utilities was found excessive, so, outputs were insufficient. Their results were extended to heterogeneous firms in Dhingra and Morrow (2011). Our extension considers a different direction - to variable technology.

5.1 Social non-optimality of equilibrium

We start this topic with defining social optimality. Under homogeneous consumers it amounts to consumer's gross utility, obtained when social planner optimizes all variables under labor balance only (first-best solution). So, assuming symmetric solution (naturally), optimality means that x^{opt}, f^{opt} and N^{opt} are the solution to optimization problem

$$\begin{cases} Nu(x) \to \max\limits_{N,x,f} \\ N(c(f)xL + f) = L \end{cases}$$

By expressing N from the labor balance and substituting, we get the simpler formulation

$$\frac{Lu(x)}{c(f)xL + f} \to \max_{x,f}$$

and come to following characterization of first- and second-order conditions.

Proposition 4. *At social optimum, FOC is*

$$\begin{cases} r_{\ln u} - r_u = \dfrac{cxL}{cxL + f} \\ c'xL = -1 \end{cases} \tag{27}$$

while SOC is

$$\mathcal{E}_c + r_u r_c \equiv r_{\ln c} - (1 - r_u)\, r_c > 0. \tag{28}$$

Moreover,

$$(1 - \mathcal{E}_c)\, \mathcal{E}_u \equiv (1 - r_{\ln c} + r_c)\, (r_{\ln u} - r_u) = 1. \tag{29}$$

Now we can compare this characterization with the equilibrium FOC obtained earlier:

$$\begin{cases} 1 - r_u = \dfrac{cxL}{cxL + f} \\ c'xL = -1 \end{cases}$$

We see that *equilibrium is optimal if and only if* $r_{\ln u} = 1$ that holds true under $u = x^\rho$, i.e., CES utility that has constant elasticity $\mathcal{E}_u \equiv \frac{xu'}{u} = \rho$. The same conclusion by Dixit and Stiglitz (1977) remains true in our setting with variable investments. Additionally, we would like to know the direction of departure from optimum. The next proposition contains such detailed comparison between optimal consumption (x^o) and equilibrium consumption (x^*), optimal investment (f^o) and equilibrium investment (f^*), optimal mass of firms (N^o) and equilibrium mass of firms (N^*).

Proposition 5. *Optimal consumption, investment, mass of firms can be bigger or smaller than related equilibrium magnitudes - depending upon increasing (IEU) or decreasing (DEU) elasticity $\mathcal{E}_u$ of utility:*

IEU: $\mathcal{E}_u' > 0 \Leftrightarrow r_{\ln u} < 1$	$\mathcal{E}_u' = 0$	*DEU:* $\mathcal{E}_u' < 0$
purchase size $x^o < x^*$	$x^o = x^*$	$x^o > x^*$
investment $f^o < f^*$	$f^o = f^*$	$f^o > f^*$
mass of firms $N^o > N^*$	$N^o = N^*$	$N^o < N^*$

Further, optimal total industry investment $(Nf)^o = N^o \cdot f^o$ is compared with equilibrium total investment $(Nf)^* = N^* \cdot f^*$, according to three different patterns:

$(1 - r_{\ln u}) \cdot (1 - r_{\ln c}) < 0$	$(1 - r_{\ln u}) \cdot (1 - r_{\ln c}) = 0$	$(1 - r_{\ln u}) \cdot (1 - r_{\ln c}) > 0$
$(Nf)^{opt} > (Nf)^*$	$(Nf)^{opt} = (Nf)^*$	$(Nf)^{opt} < (Nf)^*$

Commenting the upper table, we should say that $1 - r_{\ln u}$ is positive when utility elasticity $\mathcal{E}_u(x) = xu'(x)/u(x)$ increases in consumption. Then variety at equilibrium (the mass of firms) exceeds the optimal one and naturally, the consumption of each brand appears too small (under the opposite assumption we see the opposite departure from optimality). More importantly (that is why we mark it by red), *under DEU preferences, each firm' investment in R&D is too small because the number of firms is excessive* relative to socially-optimal one.

These DEU preferences are supposed realistic in Dixit and Stiglitz (1977), similarly bringing *insufficient* output and excessive number of firms (under exogenous technology, see also Dhingra and Morrow (2011)). Also, *insufficient* R&D investments is supported (in different, oligopoly, setting) in Dasgupta and Stiglitz (1980). We bring together these early insights in a unified framework.

Here, in contrast with previous section, IED and DED properties of preferences play no role, though CES is again the borderline between the two main patterns: IEU and DEU cases. An example of IED but DEU utility is the simple modification of CES utility: $u = (x + a)^\rho - a^\rho$ (more specific case IED but DEU is $u = x^\rho - bx$, $b > 0$).

The lower table says that total investment in market economy (Nf) is lower than socially optimal when elasticities of utility and cost display opposite monotonicity; total investment is bigger than needed when both elasticities increase or both decrease.

It is not easy to give economic interpretation of this necessary and sufficient conditions. We can only note that without studying the market there is no *ad hoc* rationale for fostering or hampering competition of varieties, and possibility of R&D investments does not change this conclusion.

5.2 Comparative statics of optimum and convergence with equilibrium

Now we study comparative statics of optimum w.r.t. market size L as we did with market equilibrium.

Proposition 6. *At social optimum the elasticities of consumption x^o, investment f^o and mass of firms N^o w.r.t. market size L are*

$$\mathcal{E}_x = -\frac{(1 - r_{\ln c})\,\mathcal{E}_c \mathcal{E}_u}{\mathcal{E}_c + r_u r_c}\,, \quad \mathcal{E}_{Lx} = -\frac{(1 - r_{\ln u})\,r_c}{\mathcal{E}_c + r_u r_c} = 1 - \frac{(1 - r_{\ln c})\,\mathcal{E}_c \mathcal{E}_u}{\mathcal{E}_c + r_u r_c}\,,$$

$$\mathcal{E}_f = -\frac{1-r_{\ln u}}{\mathcal{E}_c + r_u r_c}, \quad \mathcal{E}_{Nf} = 1 - \frac{(1-r_{\ln u})(1-r_{\ln c})\mathcal{E}_u}{\mathcal{E}_c + r_u r_c},$$
$$\mathcal{E}_N = 1 + \frac{(1-r_{\ln u}) r_c \mathcal{E}_u}{\mathcal{E}_c + r_u r_c} = \left(\frac{(1-r_{\ln c})\mathcal{E}_u}{\mathcal{E}_c + r_u r_c} - 1\right)\mathcal{E}_u\mathcal{E}_c$$

The signs of these elasticities can be divided into several patterns:

	IEU: $\mathcal{E}_u^{'} > 0$		*CEU*	*DEU:*$\mathcal{E}_u^{'} < 0$		
	$\Leftrightarrow r_{\ln u} < 1$		$r_{\ln u} = 1$	$\Leftrightarrow r_{\ln u} > 1$		
	$r_{\ln c} \leq 1$	$r_{\ln c} > 1$	$r_{\ln c} \neq 1$	$r_{\ln c} > 1$	$r_{\ln c} = 1$	$r_{\ln c} < 1$
$\mathcal{E}_{x^o}$	$\nexists$	< 0	$= -1$	< 0	$= 0$	> 0
$\mathcal{E}_{Lx^o}$	$\nexists$	< 0	$= 0$	$\in (0;1)$	$= 1$	> 1
$\mathcal{E}_{f^o}$	$\nexists$	< 0	$= 0$	> 0	$= \mathcal{E}_u \in (0;1)$	> 0
$\mathcal{E}_{N^o f^o}$	$\nexists$	> 1	$= 1$	$\in (0;1)$	$= 1$	> 1
$\mathcal{E}_{N^o}$	$\nexists$	> 1	$= 1$	$\in (0;1)$	$= 1 - \mathcal{E}_u \in (0;1)$	< 1

Here, we note that behavior of optimal investment follows three patterns, like equilibria, but now they are governed by IEU, CEU and DEU cases of preferences instead of IED, CES and DED. As in Table 1, we mark by red here the most "realistic" case for discussing innovations. This case is the IED+DEU class of utility functions. It is easy to prove that IED+DEU class includes, in particular, a sub-class of all "realistic" sums of power functions like

$$u(x) = \sum_i a_i \left((x+d_i)^{\rho_i} - d_i^{\rho_i}\right) - \sum_i b_i x^{\eta_i}: \quad a_i > 0, b_i > 0, d_i > 0, \eta_i \geq 1, \rho_i \in (0,1). \tag{30}$$

We call such coefficients $a_i, b_i, d_i, \eta_i, \rho_i$ "realistic", because they generate decreasing demand functions and concave profit functions displaying finite maximum. This sub-class includes AHARA utilities: $u(x) = x^\rho - bx$, $u = (x+d)^\rho - d^\rho$, quadratic utility $u(x) = x - ax^2$, etc. We suppose that, in spirit of Taylor decomposition, many realistic IED+DEU demands can be tightly approximated by such sums of power functions.

The IED+DEU class brings natural equilibria effects that agree with the stylized facts: equilibrium innovations increase, remaining lower than optimal ones (see Proposition 5). However, socially-optimal innovations also increase with the market size, and their convergence to equilibrium is not obvious. Yet, the proposition below finds when a growing market drives equilibrium and social optimum *closer to each other* (another optimistic conclusion of this kind we see in Dhingra and Morrow (2011) under fixed investments).

Conjecture. *Assume convex cost and utility $u(\cdot)$ of IED class (e.g., being the sum of powers as in (30). Then ratio q^*/q^o of equilibrium and optimum sizes increases monotonically up to the limit 1. Similar monotone convergence to 1 takes place for equilibrium and optimum R&D investments f^*/f^o.*[17]

This conjecture can be interpreted as a *technological benefit of a bigger country, or market integration* under realistic IED situations: not only productivity increases, but it also converges to optimal productivity. Welfare under IED case also increases as we have seen. However, when the market is not extremely big, there can be a room for governmental regulation.

5.3 Equilibrium versus optimum: governmental regulation

As we have seen, at the equilibrium the elasticity of revenue equals the elasticity of total cost $C(q)$ (including the sub-optimized $f(q)$), i.e,

$$\mathcal{E}_R(q) = 1 - r_u(x) = \mathcal{E}_C(q) \equiv xL \cdot \frac{C'(xL)}{C(xL)}.$$

[17] Similar convergence one can find between the equilibrium/optimal masses of firms $\bar{N}/\check{N}$.

Only the left-hand side of this equilibrium condition differs from our social optimality condition:

$$\mathcal{E}_u(x) = \mathcal{E}_C(q).$$

So, the natural idea for Pigouvian governmental regulation (taxing or subsidizing firms) would be *to modify the firm's revenue in such a way that it would have the same elasticity as the consumer's utility.*

To expand this idea, we note that it necessarily means some *non-linear* taxation, because linear taxation cannot modify the revenue elasticity (convexity). Namely, we are going to argue in favor of *regressive* taxation. Mathematically, we would like to find such monotone transform $G(R, x)$ of revenue (maybe, dependent on output), that everywhere, or at least in the zone of optima, the modified firm's revenue $\tilde{R}(x) = G(R(x), x)$ keeps the same elasticity as the consumer's utility:

$$\mathcal{E}_{\tilde{R}}(x) = \mathcal{E}_u(x).$$

This does not necessarily mean $\tilde{R}(x) = u(x)$, because linear transforms of G are irrelevant, there is some freedom, but not too much. The reason is that we do not allow for cross-subsidization (we abstain from the Ramsey's problem of minimizing inevitable deadweight losses across different industries through subsidization). We remain instead on the same grounds as our second-best social optimum: self-financing industry. This means imposing additional restriction

$$\tilde{R}(x) = G(R(x), x) = R(x).$$

We assume that only the revenue (not profit) can be taxed, that the government somehow knows the shape of the demand. Let us look on two "extreme" examples of such regulation in these circumstances.

Regulation example 1. Under CES utility $u(x) = x^\rho$, the elasticities of utility and revenue coincide everywhere:

$$\mathcal{E}_u(x) = \rho = \mathcal{E}_R(q) = 1 - r_u(x).$$

Thus, as well-known and confirmed by these formulae, an industry with CES utility (with iso-elastic demand) shows optimal equilibrium output, optimal innovation, and needs no regulation.

Regulation example 2. Previous example displays *unrealistically convex* demand, and the government need not stimulate innovations. Consider now a particular case of IED+DEU class of "realistic" demands: *very flat* (linear) demand generated by a quadratic utility. It is parameterized as $u(x) = a(x - \frac{1}{2b}x^2)$, $u'(x) = a(1 - \frac{1}{b}x)$, to make parameter $a > 0$ the choke-price and $b > 0$—the satiation point. Then per-consumer marginal revenue is $xu'(x) = a(x - \frac{1}{b}x^2)/\lambda$, its elasticity $\mathcal{E}_R(x) = \frac{\frac{2}{b}x}{(1-\frac{1}{b}x)}$ decreases in x twice faster than elasticity of utility $\mathcal{E}_u(x) = \frac{\frac{1}{b}x}{(1-\frac{1}{b}x)}$. Respectively, the government should practice per-consumer quantity non-linear subsidy $s(x) = \frac{a}{2b}x/\lambda$ to producer, that yields new producer's price

$$\tilde{p}(x) = p(x) + s(x) = a(1 - \frac{1}{2b}x)/\lambda.$$

As a result, the firm would be motivated to increase its output exactly to the socially optimal magnitude. To keep the industry self-financing, this subsidy should be covered by some appropriate lump-sum entry-fee from each firm starting the business (the fee is added to f_0 in firm's decisions). This regulation makes $\mathcal{E}_u(x) = \mathcal{E}_R(q)$ and thereby reduces the mass of firms from socially excessive level (see Proposition 5.1) to a socially optimal one (the reduced entry modifies the intensity of competition λ appropriately).

So far we cannot derive a general rule of such regulation more specific than $\mathcal{E}_{\tilde{R}}(x) = \mathcal{E}_u(x)$. Leaving more detailed guidelines for regulation for further study, we only stress here that under realistic IED+DEU preferences, *driving an industry closer to social optimum necessarily requires regressive taxation or progressive subsidizing of output* to reduce the number of firms and increase their outputs and R&D.

6 Exogenous cost shocks or cost subsidies

So far we considered only the impact of the market size. Instead, this section introduces the parameterized cost function $c = c(f, \alpha)$ as a decreasing function of investment and of a parameter α, and we look how the equilibrium change. The parameter can mean a technological innovation, or institutions improvement, joining WTO, subsidies for R&D, etc.—all shocks that bring cost benefits to firms through increasing the efficiency of cost-reducing investment ("process improvement"). We assume derivatives

$$\frac{\partial c}{\partial f} < 0, \qquad \frac{\partial^2 c}{\partial f^2} > 0, \qquad \frac{\partial c}{\partial \alpha} < 0, \qquad \frac{\partial^2 c}{\partial f \partial \alpha} < 0,$$

that mean decreasing (w.r.t. investment and shock) convex sub-modular marginal cost.

The equilibrium concept (and therefore the system of equilibrium equations) remains the same as under simpler costs $c = c(f)$. Moreover, Proposition 1 characterizing the equilibrium remains valid, only the notations $r_c, r_{\ln c}$ mean now

$$r_c := r_c(f, \alpha) := -\frac{\frac{\partial^2 c}{\partial f^2} \cdot f}{\frac{\partial c}{\partial f}} > 0$$

$$r_{\ln c} := r_{\ln c}(f, \alpha) := -\frac{\frac{\partial^2 \ln c}{\partial f^2} \cdot f}{\frac{\partial \ln c}{\partial f}}.$$

We are interested in the elasticities of variables x, f, N, p w.r.t. α. To formulate the result, we introduce notations of partial elasticities derived from the cost function

$$\mathcal{E}_{c/\alpha} := \frac{\partial c}{\partial \alpha} \cdot \frac{\alpha}{c} < 0$$

$$\mathcal{E}_{c'_f/\alpha} := \frac{\partial \left(\frac{\partial c}{\partial f} \right)}{\partial \alpha} \cdot \frac{\alpha}{\frac{\partial c}{\partial f}} = \frac{\partial^2 c}{\partial f \partial \alpha} \cdot \frac{\alpha}{\frac{\partial c}{\partial f}} > 0$$

Proposition 7. *Elasticities of the equilibrium variables w.r.t. cost-decreasing parameter α are*

$$\mathcal{E}_{x/\alpha} = \frac{\mathcal{E}_{c'_f/\alpha} - (1 - r_u)\, r_c \mathcal{E}_{c/\alpha}}{(2 - r_{u'})\, r_c - 1} > 0$$

$$\mathcal{E}_{f/\alpha} = \frac{(2 - r_{u'})\, \mathcal{E}_{c'_f/\alpha} - (1 - r_u)\, \mathcal{E}_{c/\alpha}}{(2 - r_{u'})\, r_c - 1} > 0$$

$$\mathcal{E}_{N/\alpha} = \frac{r'_u x}{r_u} \mathcal{E}_{x/\alpha} - \mathcal{E}_{f/\alpha}$$

$$\mathcal{E}_{Nf/\alpha} = \mathcal{E}_{\frac{p-c}{p}/\alpha} = \frac{r'_u x}{r_u} \mathcal{E}_{x/\alpha}$$

$$\mathcal{E}_{p/\alpha} = \frac{-r_u \mathcal{E}_{c'_f/\alpha} + (1 - r_u)\left((2 - r_{u'})\, r_c - 1 + r_u r_c\right) \mathcal{E}_{c/\alpha}}{(2 - r_{u'})\, r_c - 1} < 0$$

and their signs satisfy the following classification:

	DED	CES	IED
	$r'_u < 0$	$r'_u = 0$	$r'_u > 0$
$\mathcal{E}_{x/\alpha}$	> 0	> 0	> 0
$\mathcal{E}_{f/\alpha}$	> 0	> 0	> 0
$\mathcal{E}_{Nf/\alpha} = \mathcal{E}_{\frac{p-c}{p}/\alpha}$	< 0	$= 0$	> 0
$\mathcal{E}_{N/\alpha}$	< 0	< 0	?
$\mathcal{E}_{p/\alpha}$	< 0	< 0	< 0

We see here that both investment and consumption increase, price decreases in any case, whereas markups and total investments of the economy behave differently in IED and DED cases. Comparing this result with Proposition 2 we note that there the signs of elasticities of Lx, f, p w.r.t. L depend on the sign of r'_u while now in Proposition 7 similar elasticities w.r.t. α have unambiguous signs. In contrast, increasing or decreasing Nf (and markup $\frac{p-c}{p}$) w.r.t. α now depends on the sign of r'_u.

The natural conclusion from this section is that an exogenous cost-reducing shock (productivity increase) should bring higher investment and lower prices. Less trivial idea is that *cost-reducing governmental regulation* (say, tax reductions conditional on R&D investment) *could push the equilibrium closer to social optimum* in IED situation that we suppose realistic.

7 Inter-industry comparisons

Let us introduce now a parameterized utility $u = u(x, \beta)$ where $\beta > 0$ is a pro-concavity parameter (for instance, $u = x^{1-\beta}$). We have in mind inter-industry comparisons: ceteris paribus, should an industry with higher love for variety have more investment or less investment than an industry with more substitutable varieties. To formulate such general statement, we assume RLV to increase w.r.t. β:

$$r_u(x, \beta_1) < r_u(x, \beta_2)\ \forall x\ \forall \beta_1 < \beta_2, \quad i.e., \quad \frac{\partial r_u(x,\beta)}{\partial \beta} > 0, \quad i.e., \quad \mathcal{E}_{r_u/\beta}(x,\beta) = \frac{\partial r_u(x,\beta)}{\partial \beta} \cdot \frac{\beta}{r_u(x,\beta)} > 0.$$

Proposition 8. *Under increasing RLV, i.e., $\mathcal{E}_{r_u/\beta}(x,\beta) > 0$, the equilibrium R&D investment decreases w.r.t. RLV parameter β, having elasticity*

$$\mathcal{E}_{f/\beta} = -\frac{\mathcal{E}_{r_u/\beta}(x,\beta)}{(2 - r_{u'}(x,\beta)) \cdot r_c(f) - 1} < 0,$$

and all elasticities' signs satisfy the following classification:

	Increasing RLV: $\mathcal{E}_{r_u/\beta}(x,\beta) > 0$		
	$r_{\ln c} < 1$	$r_{\ln c} = 1$	$r_{\ln c} > 1$
$\mathcal{E}_{f/\beta}$	< 0	< 0	< 0
$\mathcal{E}_{q/\beta} = \mathcal{E}_{Lx/\beta} = \mathcal{E}_{x/\beta}$	< 0	< 0	< 0
$\mathcal{E}_{N/\beta}$	> 0	> 0	> 0
$\mathcal{E}_{p/\beta}$	> 0	> 0	> 0
$\mathcal{E}_{\frac{p-c}{p}/\beta} = \mathcal{E}_{\frac{Nf}{L}/\beta} = \mathcal{E}_{Nf/\beta}$	< 0	$= 0$	> 0

We see that at the equilibrium, for increasing RLV case, the investments of each firm as well as firm's size, decrease w.r.t. β, while prices of each variety as well as number of firms, increase. As to markup, total investments and total investments per capita, their behavior determines by the sign of $(r_{\ln c} - 1)$.

The natural conclusion is that *the more consumers love variety — the more firms are in the industry and the smaller is R&D investment.*

8 Multi-sector economy

An important extension is multiple industries interacting with each other under endogenous technology. We achieve it, expanding multi-industry setting from Zhelobodko et al. (2010) to endogenous technology case. Like in this earlier work, it turns out that response of prices and outputs to market size is not affected by other industries, only the mass of firms change! This surprising result in essence follows from each firm's profit maximizing combined with free entry: only the interaction of firms within the industry matter for the optimal size of the firm and optimal price policy, intra-industry relations do not matter.

Demand. Assume two industries (similarly, there can be n sectors, only the notation expands). Keeping previous notations and assumptions, we add only an unspecified upper-tier utility $U\colon R_+^2 \to R$ and additional industry with a lower-tier utility $\tilde{u}$, satisfying similar general assumptions but may be with different elasticity of substitution and DED/IED characteristics. Related consumptions $\tilde{x}$ and other variables of the second industry will also have tilde accent.

Assuming symmetric equilibrium (which can be proven to be the case), now the consumer maximizes utility in the form:

$$U(\int_0^N u\left(x_i\right) di, \int_0^{\tilde{N}} \tilde{u}\left(\tilde{x}_i\right) di) \to \max_{x,\tilde{x}} \;\; s.t.$$

$$\int_0^N p_i x_i di + \int_0^{\tilde{N}} \tilde{p}_i \tilde{x}_i di \leq 1$$

Here U the upper-tier utility function is assumed thrice continuously differentiable, strictly concave, increasing. It expresses the degree of substitution between our differentiated good (varieties) of the first sector and the second one, which can be a numeraire in particular case: $U(Nu\left(x\right), \tilde{N}\tilde{u}\left(\tilde{x}\right)) = U(Nu\left(x\right)) + \tilde{x}$.

Taking the first-order conditions we derive the inverse demand for x_i of the first sector as

$$p(x_i, \lambda) = \frac{u'(x_i)U_1'(X)}{\lambda}$$

where $X = (\int_0^N u\left(x_i\right) di, \int_0^{\tilde{N}} \tilde{u}\left(\tilde{x}_i\right) di) \in R_+^2$ is the vector of total utility from varieties and λ is the Lagrange multiplier of the budget constraint. Similar is the inverse demand in the second sector:

$$\tilde{p}(\tilde{x}_i, \lambda) = \frac{u'(\tilde{x}_i)U_2'(X)}{\lambda}.$$

On the supply side everything remains as before. Also, as before, each firm reasonable treats the general situation (X, λ) in the market parametrically, practically independent of its actions. Therefore, after combining FOC and free entry, the equilibrium condition of each firm boils down to the same equation between the elasticity of revenue and elasticity of non-linear cost:

$$\mathcal{E}_R(q/L) \equiv 1 - r_u(q/L) = \mathcal{E}_{\mathbf{C}}(q). \tag{31}$$

Thus, applying the same argument as earlier, we arrive at

Remark. *The behavior of outputs, investments and prices in each sector of the multi-sector economy respond to the market size exactly as in Propositions 2 and 3, only the masses of firms can show ambiguous effects.*

Thus, the same effects as before hold in the multi-sector economy, even the functions of price/output elasticities w.r.t. the market size are the same. In particular, when one industry demonstrates IED and another DED, simultaneously the first one decreases prices and increases R&D in response to the growing market, whereas the second one demonstrates the opposite effects!

9 Technology choice under heterogeneous firms[18]

Now we expand our analysis onto an important extension: firms having heterogeneous cost a'la Melitz (Melitz (2003)) but with variable elasticity of substitution and endogenous investment in technology.

As above, our main question is the influence of the market size (and technological parameters) upon the firm's investments in productivity, on prices and outputs, patterns of heterogeneity among firms. In contrast with homogeneous case, now we are interested not only in average variables, but in their distribution among firms also. Is it true that more efficient firms make more investments to further decrease their costs? How R&D investments in productivity change when the market size increases? (We again mean cross-countries comparison or market integration.)

As mentioned in Intro, such questions remained irrelevant under CES-utility (zero price effects), so, VES modelling is necessary here. Comparing to Zhelobodko et al. (2012), the model and conclusions are very similar, only the endogenous technological decisions are new. Comparing to our homogeneous model, we are going to find elasticity-specific patterns of market outcomes: increasing or decreasing productivity of the industry subject to the nature of preferences (IED or DED) and costs.

9.1 Model

Timing and goods. The economy includes L identical consumers, one diversified good, and continuum of potential businessmen with heterogeneous abilities, their type-parameter i being distributed according to some continuous density $\gamma(i)$ defined on $[0,\infty)$. Here higher i denotes higher cost and $\Gamma(t)=\int_0^t \gamma(i)di$ is the cumulative probability. It could be realistic to model the overlapping generations of businessmen (or business ideas) that appear and die stochastically during many periods. However, for conciseness, we use instead the one-period timing proposed by Melitz and Ottaviano (2008), which seems a good proxy for many similar periods with overlapping generations.

In the beginning of each period the whole population of businessmen is newly born without knowing their abilities. They know only the market outcome (prices, profits, etc.) of "typical" period. Among indefinitely many potential businessmen (firms), only N copies of each firm-type decide to try entering the market, and only firms with costs lower than certain *cutoff type* $\hat{i}$ survive as profitable ones (these $N,\hat{i}$ are endogenous). Each firm produces one firm-specific variety and total number of varieties thereby amounts to $N\int_0^{\hat{i}} \gamma(i)di$.

Consumers. From the consumer's point of view, all varieties bring similar satisfaction without being perfect substitutes. So, all firms with equal costs charge equal prices and get equal purchase sizes $x_{ij}=x_{ik}$. The firms will be identified only by type, skipping index j of firm's personality. Then, the problem of representative consumer (rather similar to homogeneous case) becomes

$$U=N\int_0^{\hat{i}} u(x_i)d\Gamma(i)\to\max_{x_i}\qquad \text{s.t.}\quad N\int_0^{\hat{i}} p_i x_i d\Gamma(i)=w\equiv 1,$$

where $[0,\hat{i}]$ is the set of available types of varieties, while x_i is the individual consumption of each variety of type $i\in[0,\hat{i}]$. Then, as in Section (3), the inverse demand function depends on the Lagrange multiplier λ and purchase size x_i as

$$p_i(x_i)=u'(x_i)/\lambda. \tag{32}$$

Operating firms. Each firm's individual marginal cost depends on its type identity i and its endogenous investments $f_i\geq\hat{f}$ in the form

$$\tilde{c}(f,i)=i\cdot c(f)$$

[18] This section vastly uses the approaches and proofs, notably, the cutoff behavior – from the study of Evgeny Zhelobodko, Sergey Kokovin, Mathieu Parenti and Jacques-Francois Thisse during their joint work on Zhelobodko et al. (2012), which includes only some of results. Thus, see also working paper Zhelobodko et al. (2011). As to ideas of firms' selection into R&D, we were influenced by Mrázová and Neary (2012) and by Peter Neary's presentation at the St.Petersburg Conference on "Industrial organization and spatial economics" (October 10-13, 2012).

Thereby we assume that firms are ordered by their marginal-costs functions: $\tilde{c}(f,i) > \tilde{c}(f,j) \forall f \iff i > j$ (like in Zhelobodko et al. (2012)), i.e., the family of investment functions $\tilde{c}(f,i) = i \cdot c(f)$ satisfy the Spence-Mirrlees condition (sort of super-modularity). As we have explained when comparing endogenous investment and non-linear cost, the supply side can be equivalently described by a firm-specific total cost functions

$$C(i,q,) = i \cdot c(\check{f}(q)) + \check{f}(q),$$

where $c(f)$ is the same investment function as in homogenous economy of previous section and $\check{f}$ is the sub-optimized investment, introduced in Section (3). The Spence-Mirrlees condition entails similar ordering among f-optimal total costs generated from $c(.)$: variable cost of i-th firm is simply $i \cdot c(\check{f}(q))$ so that higher types have higher costs. Naturally, all properties of functions c, C remain valid, importantly, C is concave.

The profit of an operating firm of type i is given by

$$\widetilde{\pi}(i, x_i, f_i, \lambda, L) = \left[\frac{u'(x_i)}{\lambda} - ic(f_i)\right] Lx_i - f_i.$$

The $i - th$ producer profit maximization w.r.t. x_i and f_i is equivalent to maximization w.r.t. output and gives the optimal profit function:

$$\pi(i,\lambda,L) \equiv \max_{x_i, f_i \geq \hat{f}} \left[\frac{u'(x_i)}{\lambda} - ic(f_i)\right] Lx_i - f_i = \max_{q_i} \left[\frac{u'(q_i/L)}{\lambda} q_i - C(i,q,)\right].$$

Using the FOC, the optimal x_i and f_i become functions of parameters λ and L. By substituting these $x_i(\lambda, L)$ and $f_i(\lambda, L)$ into $\widetilde{\pi}(i, x_i, f_i, \lambda, L)$, we obtain the *optimal* profit $\pi(i,\lambda,L) = \widetilde{\pi}(i, x_i(\lambda, L), f(\lambda, L), \lambda, L)$ as function of λ and L. Using this function, it is easy to obtain the condition for "boundary" or cutoff producer, i.e. such producer type $\hat{\imath}$, that her profit equals zero:

$$\pi(\hat{\imath}, \lambda, L) = 0. \tag{33}$$

All firms, operating in the market, thus have a type smaller than or equal to $\hat{\imath}$ and earn positive profits, while firms having a type exceeding $\hat{\imath}$ do not produce. It is easy to find the signs of profit partial derivatives $\frac{\partial \pi(i,\lambda,L)}{\partial i} < 0$, $\frac{\partial q^o(i,\lambda,L)}{\partial i} < 0$, $\frac{\partial \pi(i,\lambda,L)}{\partial \lambda} < 0$, $\frac{\partial \pi(i,\lambda,L)}{\partial L} > 0$ that we need below to study the solutions. In particular, L given, total differentiation of the equation (33) w.r.t. λ shows that its solution $\hat{\imath}(\lambda)$ is negative monotone.

Experimenting entrepreneurs. We have assumed that the cost functions are assigned randomly, and relative frequency of type $i \in [0, \infty)$ is determined by some distribution function $\Gamma(i)$ on $[0, \infty)$. Prior to entering the market, each entrepreneur (potential firm) does not know its actual future production cost $c(f,i)q + f + f_e$. She ad hoc spends some $f_e > 0$ - *experimenting cost* or expenditure to study one's productivity. The cost of experimenting is fixed and known to all entrepreneurs; for example, f_e is the cost of a business plan bought from a consulting firm. All firms make decisions simultaneously in Nash fashion, correctly anticipating the consumer's reactions (the demand function) and the expected competition intensity λ. This determines the equilibrium competition intensity λ such that

$$\int_0^{\hat{\mathrm{i}}} \pi(i,\lambda,L) d\Gamma(i) = f_e. \tag{34}$$

Equilibrium couple $(\hat{\lambda}, \hat{\imath})$ is defined by the system of two equations (33) and (34) with two variables, $\hat{\imath} = \hat{\mathrm{i}}(\lambda)$ and λ, other variables - outputs, investments, prices and mass N of experimenters (copies of each type) - being the consequences derived as in Section 3, but using new labor balance:

$$N\left(\int_0^{\hat{\mathrm{i}}} C(i,q_i) d\Gamma(i) + f_e\right) = L.$$

The latter equation ensures that economy is closed, all labor is spent for production and experimenting, and zero expected profit (34) shows that only labor makes consumer's income. Hence, the mass of operating firms is given by $N\Gamma(\hat{i}) \leq N$ and $\hat{i}, N$ are determined by free entry.

Note, that this equilibrium definition builds on the assumption that all firms do invest something in R&D ($f > f_0$) and that there is only one cutoff, i.e., all types $i > \hat{i}$ do produce. Strictly speaking, both properties should be proven, there could be different equilibria, to be studied in another paper. The second property can be checked by the technique proposed in Mrázová and Neary (2012) through ensuring super-modularity of profit.

9.1.1 Existence and uniqueness

For **equilibrium existence and uniqueness**, we note that, by implicit function theorem, optimal profit $\pi(i,\lambda,L)$ decreases w.r.t. i (due to cost super-modularity), decreases w.r.t.λ, increases w.r.t. L. Therefore, equation (33) determines the cutoff type $\hat{i} = \hat{i}(\lambda)$ as a decreasing function of λ. Thereby, the left-hand side in (34) is decreasing in λ for two reasons: decreasing upper limit of integration and decreasing integrand $\pi(i,\lambda,L)$. This gives uniqueness of equilibrium $\hat{\lambda}$, if it exists and optimal profits $\pi(i,\lambda,L)$ are single-valued. To see existence, note that for each i profit goes from $\lim_{\lambda\to 0}\pi(i,\lambda,L) = \infty$ to $\lim_{\lambda\to\infty}\pi(i,\lambda,L) \leq 0$. So, the integral must reach positive value f_e somewhere. The only obstacle for existence may be nonexistence of optimal profits $\pi(i,\lambda,L)$ per se. However, existence condition imposed in Section 3 for homogenous economy ensure equilibrium existence for heterogeneous firms also.

9.2 Comparisons between good and bad firms

Monotonicity of profit and optimal output q_i w.r.t. type ($\frac{\partial\pi(i,\lambda,L)}{\partial i} < 0$, $\frac{\partial q^o(i,\lambda,L)}{\partial i} < 0$) enables to immediately formulate the monotonicity properties of the equilibrium curves of outputs, prices and markups.

Proposition 9. *At a given equilibrium, efficient firms have higher outputs, more consumers and smaller prices:*

$$i < j \;\Rightarrow\; q_i > q_j,\; x_i > x_j,\; p_i < p_j.$$

Equilibrium curve of markups decreases w.r.t. cost-type i in IED case ($i < j \;\Rightarrow\; M_i > M_j$), increases in DED case ($i < j \;\Rightarrow\; M_i > M_j$) and remains constant $(1-\rho)$ if $u(x) = x^\rho$ (CES).

Proof: see Zhelobodko et al. (2012), and apply our conversion of endogenous investment to non-linear costs.

This proposition shows again how peculiar is CES assumption making firms' markups independent of their productivity. Furthermore, DED case show theoretical possibility of a paradoxical high markups for inefficient firms. Indeed, when outputs decrease too fast in cost, the possible strategy for small firms is to compensate low output with too high markup, whereas better firms use output-expanding strategy. To support or falsify existence of such industries is an empirical question.

9.3 Growing market size and productivity

For **comparative statics** w.r.t. population L we can study the second equation. Following Zhelobodko et al. (2012) and using partial derivatives mentioned ($\frac{\partial\pi(i,\lambda,L)}{\partial\lambda} < 0$, $\frac{\partial\pi(i,\lambda,L)}{\partial L} > 0$), it is easy to show that equilibrium intensity of competition $\hat{\lambda}$ increases with L, i.e.

$$\frac{d\hat{\lambda}(L)}{dL} > 0.$$

This property allows us to find important changes in equilibrium value of the cut-off firm $\hat{i}(L)$ w.r.t. market size L. Thus, expanding Proposition 3 of Zhelobodko et al. (2012) to concave non-linear costs (avoided in Zhelobodko et al. (2012) but crucial here because of $C(.)$ generated from $c(.)$) we formulate now main comparative statics result under heterogeneity. It holds locally at the equilibrium $\bar{x}$ and globally when condition IED or DED is global.

Proposition 10. *Equilibrium intensity of competition $\hat{\lambda}$ increases with market size. The cutoff type $\hat{i}$ decreases under IED ($r_u'(\bar{x}) > 0$), increases under DED ($r_u'(\bar{x}) < 0$), remaining constant when utility $u(\cdot)$ is the CES.*

Proof: see Zhelobodko et al. (2012), and apply our conversion of endogenous investment to non-linear costs.

So, in IED case cutoff $\hat{i}$ value decreases with L, and thereby some group of least-efficient firms leaves the market. Thereby, the average *productivity should increase*, if generally outputs increase (see an example below). Then two pro-efficiency forces work together: each firm expands R&D and the firms' composition improves, simply because *worst firms drop out of the market*.

This joint effect can be an important reason for gains from international trade (markets integration), underestimated by formal theory so far. In contrast to typical views of the journalists, when domestic weakest firms do not survive, this should have a positive productivity effect, at least in the long run. Interestingly, DED case shows a theoretical possibility of the opposite effect for related industries. Again, CES case looks very peculiar and inappropriate for modelling productivity changes in response to market size.

Further, we would like to get similar predictions about outputs and prices. For the case of constant marginal cost, working paper Zhelobodko et al. (2011) derives such comparative statics. Namely, increasing market size *drives up outputs of all existing firms, and all prices go down in IED market, whereas effects are opposite in DED market.* Naturally, CES market (original Melitz model) shows no impact.

Obviously, these effects *remain valid in our more general model when all firms operate in constant-cost regime* (minimal investment). Besides, one would expect somewhat similar behavior under weakly changing marginal costs of each firm. Other general analytical answers on changes in outputs and prices are hardly attainable. They dependent on the distribution. So, we turned to special cases.

Special case 1: CES investment function. For general case of our complicated model with investments we do not hope for general *analytical* conclusions of this kind. So, we turned to studying special functional forms. On the cost side, a form allowing for clear predictions is CES investment function with some power $\gamma > 0$[19]

$$c(f,i) = \begin{cases} i \cdot f^{-\gamma} & f > \hat{f} \\ i \cdot \hat{c} & f \leq \hat{f} \end{cases} \tag{35}$$

It is easy to check, that this investment function generates total cost C with constant elasticity in the R&D regime (though it increases in another regime):

$$\mathcal{E}_C(q) = \frac{1}{1+\gamma} : \quad f > \hat{f}.$$

Then, using main equilibrium equation

$$E_R(q/L) = 1 - r_u(x) = \mathcal{E}_C(q) \tag{36}$$

(where R is revenue) applied only to the cutoff firm, it is easy to derive the impact of market size for such special costs. Assume IED market ($r_u' > 0$) and R&D regime for all firms. Since $E_R(q_i/L)$ decreases and $E_R(q_i/L) = \frac{1}{1+\gamma}$ remains constant, the markup $r_u(x_i)$ of the current cutoff firm (changing the identity i) remains constant, so, its output remains constant.

[19]CES investment function is incompatible with CES utility and DED utilities when modeling R&D regime.

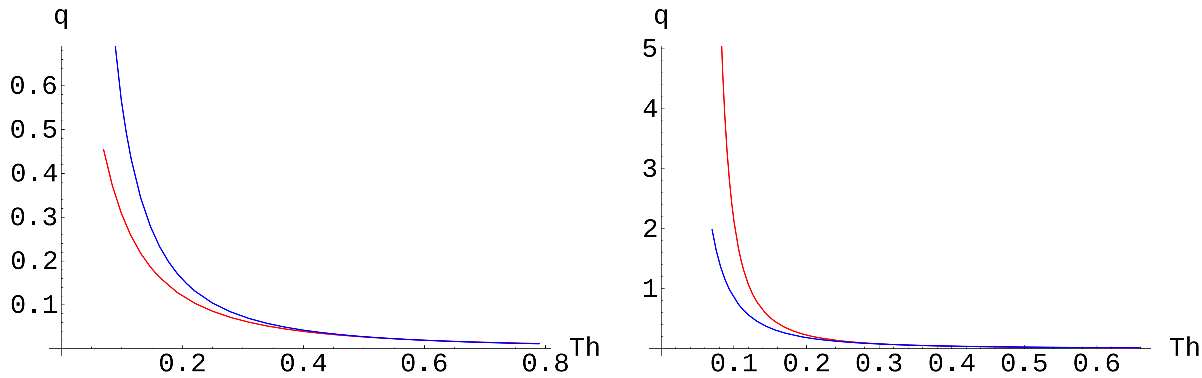

Figure 3: (A) Outputs under IED as function of type; (B) Outputs under DED.

Special case 2: AHARA utility function and HARA investment function. Instead of special costs we can study special utilities. Our favorite functional class is augmented hyperbolic average risk aversion (AHARA) utility, formulated as

$$u(x) = \frac{(d+x)^{\rho} - d^{\rho}}{h} - l \cdot x : \quad \rho \in (0,1),\, d \geq 0,\, l \lesseqgtr 0,\, h > 0.$$

The motive for this functional form is to have CES as a special case under $d = 0$, $l = 0$ and to model both effects: (1) IED effect under $d > 0$ or/and $l < 0$; (2) DED effect under $l > 0$. AHARA is a sufficiently rich functional class. In particular, it tightly approximates a linear demand when both d and l tend to infinity, being compensated by sufficiently small h. Therefore, changing the parameters, we cover the whole interval of IED cases between CES and linear demand.

Thus, we turned to massive simulations for this parameterized functional classes. “Massive” means exploring *whole domain* of possible parameters through thousands of points picked from the domain. Then we perceive the results as sort of theorems.

Namely, we used the investment function

$$c(f,i) = \begin{cases} i \cdot (a + f^{-\gamma}) & f > \hat{f} \\ i \cdot \hat{c} & f \leq \hat{f} \end{cases}$$

and AHARA function with $\rho = 1/2$, $d = 0$, and parameter varying in $l \in [-10, 10]$ with step 0.01. Simulations has shown the following results.

Observation 1 (Outputs, investments, markups, welfare). *Suppose market increases from L_1 to L_2. Then, at these two equilibria compared:*

(i) In IED case, there is an unaffected type i_0 keeping its output unchanged, whereas all more productive firms increase their outputs ($i < i_0 \Rightarrow q_i(L_1) < q_i(L_2)$) and all worse firms $i > i_0$ decrease outputs ($i > i_0 \Rightarrow q_i(L_1) > q_i(L_2)$); all prices go down.

(ii) In DED case, there is an unaffected type i_0 keeping its output unchanged, whereas all more productive firms increase their outputs ($i < i_0 \Rightarrow q_i(L_1) > q_i(L_2)$) and all worse firms $i > i_0$ decrease outputs ($i > i_0 \Rightarrow q_i(L_1) < q_i(L_2)$); all prices go up.

In all cases, investments and markups change in the same direction as outputs. Welfare unambiguously increase in IED case.

To illustrate this observation, in Figure 3 we draw related behavior of outputs-curve and prices-curve in IED (left panel) and DED (right panel) cases. The red curve shows initial characteristics of all firms, whereas the blue one shows similar curve after the market size increased from L_1 to L_2. The behavior of outputs and prices for IED (left panel) and DED (right panel) is illustrated by Figure 4.

Similar patterns of changes in outputs and prices were found in Zhelobodko et al. (2011) for constant marginal costs. Now we expand them to special cases of non-linear concave costs generated from the

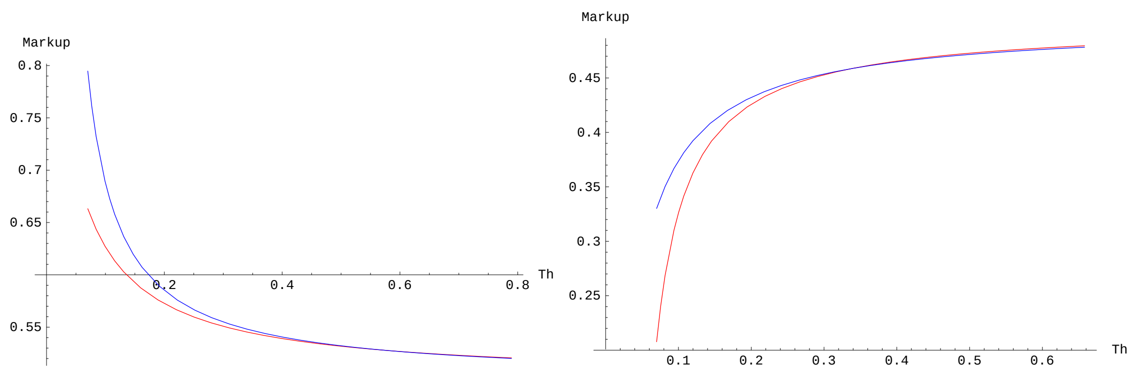

Figure 4: (A) Markups under IED as function of type; (B) Markups under DED.

investment function and find investments. Theoretically, we expect that more exotic shifts of parameters than we tried may find different patterns.

We interpret the mechanism of these changes stated in Proposition 10 and Observation 1 as follows. When market increases, existing firms start making more profits. This invites new experimenting businessmen ($N \uparrow$) and thereby pulls up consumer's marginal utility of income, that can be spent on more varieties ($\lambda \uparrow$). The two countervailing forces changing the inverse demand for each variety is market size pulling its quantity-dimension to the right ($L \uparrow$) and competition intensity, pushing its price-dimension down ($\lambda \uparrow$). Among all firms there exists a firm i_0 where these forces exactly outweigh each other in their influence on markup, output and investment respectively. However, to the right and to the left from this firm changes are very different in IED or DED markets. In IED case that appears economically more probable, these are productive firms who increase their output, investment and markups, whereas the share of weak and negatively affected firms looks very small. Average output goes up and *average productivity improves for 3 reasons*: (1) efficient firms start producing more; (2) the cutoff (threshold between producing and idle firms) goes down, i.e., worst firms decrease their output and even exit the market; (3) better firms further increase their investments in productivity.

Unfortunately for DED markets, their productivity effect is negative, since the same 3 reasons change the sign; here most firms decrease their outputs compensating this by charging higher markups, that potentially may be harmful for welfare.

10 Conclusions and extensions

Endogenous technology choice is a popular topic, but only recently has its theoretical representation been achieved in a rich enough model (Vives (2008)), one demonstrates that both the positive and negative impacts of a big market on investments in productivity.

Having in mind extensions to trade and cross-countries comparisons, we modify the Vives's model into the monopolistic competition framework, get rid of the quasi-linearity assumption (absent income effect) and strategic oligopolistic considerations. The resulting model becomes simpler and more tractable.

The findings include necessary and sufficient conditions for positive (under increasingly-elastic demand – IED) and negative (under decreasingly-elastic demand) effects of market expansion onto R&D investments. Further, these investments can be excessive or insufficient from the social optimality perspective, the outcome depends on increasingly or decreasingly elastic utility. The most plausible case is the combination IED+DEU properties; then equilibrium R&D investments are insufficient but increase and become closer to the (increasing) social optimum, i.e., again *productivity enhances*. Certain governmental regulation may help in the same direction, either in the form of taxing/subsidizing the revenue, or R&D costs.

An important extension is multiple industries interacting with each other under endogenous technology:

the main findings about R&D remain valid, because the industries interect only through the marginal utility of income.

Among the extensions studied, heterogeneous firms is the most interesting one. Here the main distinction between the IED and DED markets remains, and in the first (more plausible) case the firms also generally show increasing R&D in response to market expansion, *productivity enhances*. More importantly, in IED case the cutoff decreases, that means the second important way of enhancing productivity: *better firms selection* by the market.

We hope that our approach sets interesting questions for empirical studies: Do markets of various goods really differ in their increasing or decreasing elasticity of substitution and related market effects? Do bigger markets (countries or cities) have higher R&D investments in certain industries, unlike other industries?

For policy-making, our topic may be interesting because of new understanding of gains from trade: technological changes in response to trade liberalization. Furthermore, for modernization and active industrial policy practiced in some countries it can be interesting, which equilibrium outcome in various sectors may follow from some stimulating measures like tax reductions conditional on R&D.

Possible extensions of our model include introducing detailed international trade, though we expect there similar productivity effects as in our basic model, because lowering trade barriers work similarly to market expansion.

11 Appendix A: Lemmas on comparative statics without differentiability

Like in our another paper, Gorn et al. (2012), devoted to fixed technologies, we need the properties of the maximal per-consumer profit $\pi_u^*(\lambda)$ and the (set-valued) argmaximum

$$X_u^*(\lambda) \equiv \arg\max_{x \in \mathbb{R}_+} \pi(x, \lambda) = \arg\max_{x \in \mathbb{R}_+} u'(x)Lx/\lambda - C(Lx) : \tag{37}$$

which (under $\lambda > 0, L > 0$) is equivalent to maximizing "normalized" profit:

$$X_u^*(\lambda) = \arg\max_{x \in \mathbb{R}_+} \lambda\pi(x, \lambda)/L = \arg\max_{x \in \mathbb{R}_+} u'(x)x - \lambda C(Lx)/L.$$

Geometrically, comparative statics of $\arg\max_{x \in \mathbb{R}_+} \pi(x, \lambda)$ means that revenue curve $R_u(x) \equiv u'(x)Lx$ decreases in λ, i.e., the left-hand side decreases in the FOC equation

$$LR_u(x)/\lambda = LC'(Lx).$$

Based on ideas of Milgrom and co-authors, we define three kinds of "decreasing" mapping (set-valued function) $X : \mathbb{R} \to 2^{\mathbb{R}}$. We call a mapping $X(\lambda)$ (strictly) *decreasing*, when its extreme members decrease in the sense

$$\bar{\lambda} > \tilde{\lambda} \Rightarrow \min_{x \in X(\bar{\lambda})} < \min_{\tilde{x} \in X(\tilde{\lambda})} \quad \text{and} \quad \max_{x \in X(\bar{\lambda})} < \max_{\tilde{x} \in X(\tilde{\lambda})}, \tag{38}$$

and *non-increasing* when all the inequalities are non-strict. We call X *strongly decreasing*, when all its selections decrease in the sense

$$\bar{\lambda} > \tilde{\lambda} \Rightarrow \bar{x} < \tilde{x}, \forall \bar{x} \in X(\bar{\lambda}) \, \forall \tilde{x} \in X(\tilde{\lambda}). \tag{39}$$

The latter (strongest) version of negative monotonicity implies mapping X single-valued almost everywhere but for isolated points (downward jumps).

To prove the strongest type of monotonicity of our argmaxima X_u^* w.r.t. (λc), we shall apply three known lemmas and derive the fourth lemma from them, more closely related to what we need. The first is the following simplified version of a theorem from Milgrom and Roberts (1994, Theorem 1).[20] It predicts strictly monotone comparative statics of both extreme roots $\hat{x} \le \check{x}$ of any equation $g(x, \lambda) = 0$ with a parameter λ.

Lemma 1. (Monotone roots, Milgrom and Roberts): *Assume a partially ordered set Λ, some $\overline{x} > \underline{x}$ and a parameterized function $g(.,.) = g(x, \lambda) : [\underline{x}, \overline{x}] \times \Lambda \to \mathbb{R}$ which is continuous and weakly changes the sign, in the sense $[g(\underline{x}, \lambda) \ge 0$ & $g(\overline{x}, \lambda) \le 0 \ \forall \lambda \in \Lambda]$. Then for all $\lambda \in \Lambda$:*

(i) there exist some non-negative root(s) of equation $g(x, \lambda) = 0$, including the lowest solution $\hat{x} \equiv$ $\sup\{x | g(x, \lambda) \ge 0$ *and the highest solution $\check{x} \equiv \inf\{x | g(x, \lambda) \le 0\}$;*[21]

(ii) if our function $g(x, \lambda)$ is non-increasing w.r.t. λ everywhere, then both extreme roots $\hat{x}(\lambda)$, $\check{x}(\lambda)$ are non-increasing w.r.t. λ, i.e., $\bar{X}(\lambda)$ is non-increasing;

(iii) if, moreover, $g(x, \lambda)$ is decreasing in λ and strictly changes the sign $[g(\underline{x}, \lambda) > 0$ & $g(\overline{x}, \lambda) <$ $0 \ \forall \lambda \in \Lambda]$, then both extreme roots $\hat{x}, \check{x}$ are decreasing, i.e., $\bar{X}(\lambda)$ decreases.

The intuition behind this lemma is simple: when we shift down any continuous curve whose left/right wings are above/below zero—the roots should decline. More subtile fact is that when some root $\check{x}$ disappears, the jump goes in the same direction as all continuous changes, i.e., downward.

We apply this lemma to the (continuous) auxiliary function g gained from FOC of $\pi(x, \lambda)$:

$$\bar{X}(\lambda) \equiv \{x | \, g(x, \lambda) \equiv [u'(x) + xu''(x) - \lambda \cdot C'(Lx)/L] = 0\}, \tag{40}$$

using $\Lambda = [0, \infty)$. We conclude that mapping $\bar{X}$ is "non-increasing". We would like to enforce this property; to find "decreasing" $\bar{X}$ at those λ and domains $[\underline{x}, \overline{x}]$, where we can apply claim (iii). Locally, this task is easy: at a given λ, we can apply (iii) to any vicinity $(\underline{x}, \overline{x}) \ni \acute{x} > 0$ of any positive local argmaximum $\acute{x} | g(\acute{x}, \lambda) = 0$—whenever strict SOC holds. The latter means that $u'(x) + xu''(x)$ decreases at $\acute{x}$, i.e., isolated argmaximum. Thereby, *any positive local argmaximum x satisfying strict SOC—locally decreases in λ.*

Searching more globally for decreasing $\bar{X}$, on a positive ray we would like to identify a subinterval $(\lambda_{min}, \lambda_{max}) \subset [0, \infty)$ where claim (iii) is applicable. This amounts to finding where all roots of equation (40) are positive and finite, under Assumption 1.

Lowest λ yielding $x \in (0, \infty)$. Consider the case when our elementary revenue $R_u(x) = xu'(x)$ has a finite global argmaximum denoted

$$x_{max} \equiv \arg\max_x R_u(x)$$

(that implies satiable demand). Then, obviously, all positive λ enable *finite* solutions to (40), i.e., we must take the lower bound $\lambda_{min} = \underline{MR} = u'(x_{max}) + x_{max} u''(x_{max}) = 0$ (using notations from (3)). Similar is the result under insatiable demand ($x_{max} = \infty$) but zero limiting value $\lim_{x \to \infty}(u'(x) + xu''(x)) = 0$, using our assumptions. Anyway, we must take zero $\lambda_{min} = \underline{MR} = 0$ when we search for an interval $(\lambda_{min}, \lambda_{max})$ bringing positive finite roots of g.

Highest λ yielding $x \in (0, \infty)$. Recall notation $\overline{MR}$ from (3) and consider the case of finite derivative at the origin ($\overline{MR} < \infty$), that implies chock-price. Then all high parameters $\lambda \ge \overline{MR}$ should bring zero solutions $\hat{x}(\lambda) = \check{x}(\lambda) = 0$ to (40), for lower parameters the solutions are positive. In the case of infinite derivative $\overline{MR} = \infty$ all λ bring positive x. We conclude that anyway we must take finite or infinite $\lambda_{max} = \overline{MR}$ as a boundary, that determines the open interval

$$\hat{\Lambda} \equiv (\lambda_{min}, \lambda_{max}) \equiv (0, \overline{MR}),$$

which brings positive finite roots of g.

Now we can apply claim (iii) to this interval $\hat{\Lambda}$, because our function $g(x, \lambda) \equiv [u'(x) + xu''(x) - \lambda]$ takes positive value $\overline{MR} - \lambda > 0$ at the lower boundary $\underline{x} = 0$ and negative value $\underline{MR} - \lambda < 0$ at $\overline{x} = x_{max}$

[20] Their original Theorem 1 uses $g(x, t)$ *non-decreasing* in t, continuous "but for upward jumps", and domain $[\underline{x}, \overline{x}] = [0, 1]$ which makes a minor difference.

[21] Naturally, when finite $\hat{x}(\lambda) = \min\{x | g(x, \lambda) = 0\}$, $\check{x}(\lambda) = \max\{x | g(x, \lambda) = 0\}$).

(for all $\lambda \in \hat{\Lambda}$). Additionally, g remains strictly decreasing in λ. It also strictly decreases in x at both boundaries $(\underline{x}, \overline{x})$, because of strict concavity of $xu'(x)$ at 0 and x_{max} (Assumption 1). Thus, our function $g(x, \lambda)$ satisfies the boundary conditions and monotonicity conditions needed for Lemma 1-(iii), which implies *strict decrease of the extreme roots* $\hat{x}(\lambda) \leq \check{x}(\lambda)$ on $\hat{\Lambda}$.

It must be added that both extreme roots $\hat{x}(\lambda) \leq \check{x}(\lambda)$ of (40) are the local maxima (not minima) of function $\pi(x, 1, \lambda) \equiv xu'(x) - \lambda x$, because of SOC. Indeed, by definition of $\hat{x}, \check{x}$, function $g(x, \lambda) > 0$ must be (strictly) decreasing in some left vicinity of the left point $\hat{x}$, and in some right vicinity of $\check{x}$. Using continuous differentiability of $xu'(x)$ (Assumption 1) we expand this decrease to complete (left and right) vicinities of each point $\hat{x}, \check{x}$. This decrease of $g(x, \lambda) \equiv \pi'(x, 1, \lambda)$ means SOC. We can summarize our arguments as follows.

Proposition (Monotone local argmaxima). *Each local argmaximum of the normalized profit* $\lambda\pi(x, \lambda)/L$ *is non-increasing w.r.t.* $\lambda \geq 0$. *Moreover, the argmaximum decreases when being positive and finite, which is guaranteed only on interval* $\hat{\Lambda} \equiv (0, \overline{MR})$. *In the case of (very big) finite* $\lambda \in [\overline{MR}, \infty)$ *all argmaxima are zero.*

Now, to establish similar monotonic behavior of *global* argmaxima set X_u^* we use "single crossing" notion and Theorems 4, 4′ from Milgrom and Shannon (1994) simplified here for our case of real parameter t and unidimensional real domain $S(t)$ of maximizers.

Consider a function $g : \mathbb{R}^2 \to \mathbb{R}$. If $g(x', t'') \geq g(x'', t'')$ implies that $g(x', t') > g(x'', t') \, \forall (x' > x'', t' > t'')$, then g satisfies the *strict single crossing* property in $(x; t)$. Similarly, *single crossing* property means

$$[g(x', t'') \geq g(x'', t'') \Rightarrow g(x', t') \geq g(x'', t') \, \forall (x' > x'', t' > t'')$$

and

$$g(x', t'') > g(x'', t'') \Rightarrow g(x', t') > g(x'', t') \, \forall (x' > x'', t' > t'')]$$

(essentially, in these two versions of single-crossing notion, parameter t strictly or weakly amplifies monotonicity of g in x, alike supermodularity).

Lemma 2 (Monotone argmaxima, Milgrom and Shannon). *Consider a domain* $S(t) : \mathbb{R} \to 2^{\mathbb{R}}$ *which is non-shrinking w.r.t. t (non-decreasing by inclusion) and a function* $g : \mathbb{R}^2 \to \mathbb{R}$. *If g satisfies the single crossing property in* $(x; t)$, *then* $\arg\max_{x \in S(t)} g(x, t)$ *is monotone non-decreasing in t. If g satisfies the strict single crossing property in* $(x; t)$, *then every selection* $x^*(t)$ *from* $\arg\max_{x \in S(t)} g(x, t)$ *is monotone non-decreasing in t.*

It is important that the latter claim about all selection implies that all points of multi-valued $g(., t)$ are "isolated", in the sense that there is no *open* interval of multi-valuedness (we could claim "solid" single-valuedness but such enforcement is not necessary). The third result that we need—is similar to envelope Theorems 1, 2 from Milgrom and Segal (2002) but for more trivial conditions and claim, not needing special proof.

Lemma 3 (Monotone maxima) *Consider a compact choice set X, continuous function* $g(x, t) : X \times [0, 1] \to \mathbb{R}$ *and* $\pi^*(t) = \sup_{x \in X} g(x, t)$. *If* $\pi(x, t)$ *continuously decreases in t for all* $x \in X$, $t \in (0, 1)$, *then its maximal value* $\pi^*(t)$ *continuously decreases in* $t \in (0, 1)$.

Now, using our notations $\underline{MR}, \overline{MR}$, x_{max} and new notions

$$\lambda_{cmax} \equiv \overline{MR}/LC'(Lx_{max}), \;\; R_{max} \equiv x_{max}u'(x_{max}),$$

(that can be infinite) we formulate and prove the result we were long driving to.

Lemma 4 (Monotone argmaxima and maxima): *Consider some given* $L > 0$ *and parameter* λ *increasing in the open interval* $\Lambda(L) = (0, \lambda_{cmax})$. *Then:*

(i) On $\Lambda(L)$ *the argmaxima set* $X_u^*(\lambda) \equiv \arg\max_x [xu'(x) - \lambda C(Lx)/L]$ *is non-empty and strongly decreases from* x_{max} *to 0 (i.e., all its selections decrease).*

(ii) The maximal objective function $\pi_u^*(\lambda) \equiv [Lxu'(x)/\lambda - C(Lx)]$ *continuously decreases from* $+\infty$ *to 0. Outside* $\Lambda(L)$, *for bigger* $\lambda \geq \lambda_{cmax}$ *all argmaxima and maxima remain zero, whereas the limiting values are:*

$$\lim_{\lambda\to 0}\pi_u^*(\lambda)=\infty,\ \lim_{\lambda\to\infty}\pi_u^*(\lambda)=0. \tag{41}$$

Proof. It is convenient to argue about auxiliary profit $g = xu'(x) - \lambda C(Lx)/L$ and then return to real one.

Using $S \equiv \mathbb{R}_+$, we can apply Lemma 2 to our auxiliary function $g(x,t) \equiv \pi(x,-t)$ with $\lambda = -t > 0$ because evident is strict single crossing property: increase of $\pi_x'(x,-t) = u'(x) + xu''(x) + t$ w.r.t. $t < 0$. Thereby, whenever $X_u^*(\lambda)$ exists, *every selection* $x^*(\lambda)$ from $X_u^*(\lambda)$ is monotone *non-increasing* when $x > 0$. This yields almost unidimensional $X_u^*(\lambda)$, i.e., absence of any open intervals for λ maintaining multi-valued $X_u^*(\lambda)$. In other words, $X_u^*(\lambda)$ is single-valued but for some "isolated" downward jumps. In essence, this fact follows from smoothness of $xu'(x)$ (Assumption 1). Smoothness makes function $\pi_x'(x,-t)$ single-valued and strict single crossing property applicable (geometrically, a smooth set—undergraph of $xu'(x)$—cannot have multiple tangent slopes λ at the same point x).

To transform such monotonicity into *strongly decreasing* $X_u^*(\lambda)$ on interval $\Lambda(L)$ (at finite positive X_u^*), we apply Proposition 1 used for any local maximum. Since global maxima are among local ones, in the intervals of single-valued $X_u^* = \hat{x} = \check{x}$ they must strictly decrease. The remaining isolated points of multi-valued X_u^* are the points of *downward* jumps, as we have found. We conclude that mapping $X_u^*(\lambda)$ *strongly* decreases on interval $\Lambda(L)$, remaining infinite for smaller λ and remaining zero for higher λ.

Now we turn to the value function and apply Lemma 3 to ensure monotonicity of maximal $\pi_u^*(\lambda)$.[22] Indeed, the objective function $\pi_u(x,\lambda)$ continuously decreases w.r.t. λ everywhere under positive x. Thereby its optimal value π_u^* also continuously decreases when positive, i.e., on our interval $(0,\lambda_{cmax})$. The optimal value $\pi_u^* \to 0$ when $\lambda \to \lambda_{cmax}$ because of monotonicity and zero lower bound found in proposition 1, so continuity at the upper boundary of our interval $\Lambda(L)$ is maintained. Similar logic proves continuity at the lower boundary $\underline{MR} = 0$. Thus, the maximal value of $\pi_u^*(\lambda)$ decreases *continuously* from $\overline{MR}$ to 0 under increasing $\lambda \in [0,\infty)$. The transfer of statements from auxiliary profit $g(\lambda)$ to initial $\pi_u^*(\lambda)$ is rather obvious. This completes the proof of lemma.

12 Appendix B: Proofs

12.1 Proof of Proposition 1

In symmetric case, the equilibrium equations (5), (6), (9), (10) are

$$\frac{u''(x)x + u'(x)}{\lambda} - c(f) = 0 \tag{42}$$

$$c'(f)Lx + 1 = 0 \tag{43}$$

$$\frac{u'(x)}{\lambda} - c(f) = \frac{f}{Lx} \tag{44}$$

$$N\left(c(f)xL + f\right) = L. \tag{45}$$

Let us rewrite (42) as

$$(1 - r_u(x)) \cdot \frac{u'(x)}{\lambda} - c(f) = 0. \tag{46}$$

Substitute (44) in (46), one has

$$\frac{r_u(x)}{1 - r_u(x)} = \frac{f}{Lxc(f)} \tag{47}$$

[22] For revealing monotonicity of $\pi_u^*(\lambda, c)$ we cannot use more standard envelope theorem since π_u^* appears non-differentiable at the asymmetry points.

Hence (11) is shown. Further, using (43) one has from (11)

$$(1 - \mathcal{E}_c(f))\,(1 - r_u(x)) = 1$$

Now to obtain (12) it is sufficient to remark that

$$r_{\ln c}(f) = -\frac{\left(\dfrac{c'(f)}{c(f)}\right)'}{\dfrac{c'(f)}{c(f)}} \cdot f = r_c(f) + \mathcal{E}_c(f). \tag{48}$$

Further, in symmetric case, (8) is

$$-\frac{(u'''(x)x + 2u''(x))\,c''(f)x}{\lambda} - \left(c'(f)\right)^2 > 0,$$

i.e.

$$-\,(2 - r_{u'}(x)) \cdot \frac{u''(x)c''(f)x}{\lambda} - \left(c'(f)\right)^2 > 0. \tag{49}$$

One has due to (42), (44) and (43)

$$\frac{u''(x)x}{\lambda} = c(f) - \frac{u'(x)}{\lambda} = -\frac{f}{Lx} = c'(f)f.$$

Hence (49) is

$$-\,(2 - r_{u'}(x))\,c'(f)c''(f)f - \left(c'(f)\right)^2 > 0,$$

i.e.

$$((2 - r_{u'}(x))\,r_c(f) - 1)\left(c'(f)\right)^2 > 0.$$

Now to obtain (13) it is sufficient only to remark that $r_u(x) < 1$ due to, for example, (11).

Finally, (14) is obvious due to (45).

As to the the price, since in symmetric equilibrium (4) is

$$p = \frac{u'(x)}{\lambda}$$

and from (46)

$$\frac{u'(x)}{\lambda} = \frac{c(f)}{1 - r_u(x)}$$

one has (15).

As to the markup, due to (15) one has

$$\frac{p - c(f)}{p} = r_u(x) \tag{50}$$

Let us express $r_u(x)$ in terms of N, f and L. One has due to (47)

$$r_u\,(x) = \frac{f}{f + Lc(f)x}$$

hence, due to (45)

$$r_u\,(x) = \frac{Nf}{L} \tag{51}$$

Now from (50) and (51) one has (16).

12.2 Proof of Proposition 2

Let us first prove (20), (22) and (24), i.e. calculate $\mathcal{E}_x$, $\mathcal{E}_f$ and $\mathcal{E}_N$, using (11), (43) and (45), i.e.

$$\frac{r_u(x)\,x}{1-r_u(x)} - \frac{f}{Lc(f)} = 0 \tag{52}$$

$$c'(f)Lx = -1 \tag{53}$$

$$(c(f)xL + f)\,N - L = 0 \tag{54}$$

Let us calculate the total derivatives w.r.t. L:

$$\left(\frac{r_u(x)\,x}{1-r_u(x)} - \frac{f}{Lc(f)}\right)'_L + \left(\frac{r_u(x)\,x}{1-r_u(x)} - \frac{f}{Lc(f)}\right)'_x \cdot \frac{\partial x}{\partial L} + \left(\frac{r_u(x)\,x}{1-r_u(x)} - \frac{f}{Lc(f)}\right)'_f \cdot \frac{\partial f}{\partial L} = 0$$

$$\left(c'(f)Lx\right)'_L + \left(c'(f)Lx\right)'_x \cdot \frac{\partial x}{\partial L} + \left(c'(f)Lx\right)'_f \cdot \frac{\partial f}{\partial L} = 0$$

$$((c(f)xL+f)\,N-L)'_L + ((c(f)xL+f)\,N-L)'_x \cdot \frac{\partial x}{\partial L} + ((c(f)xL+f)\,N-L)'_f \cdot \frac{\partial f}{\partial L} +$$

$$+ ((c(f)xL+f)\,N-L)'_N \cdot \frac{\partial N}{\partial L} = 0$$

i.e., in terms of elasticities $\mathcal{E}_x$, $\mathcal{E}_f$, $\mathcal{E}_N$, using the identity

$$r'_u(x)\,x = (1 + r_u(x) - r_{u'}(x))\,r_u(x)$$

and equations (53), (54) (52), we obtain

$$\frac{f}{L^2c(f)} + \frac{2-r_{u'}(x)}{1-r_u(x)} \cdot \frac{f}{L^2c(f)} \cdot \mathcal{E}_x + \left(\frac{c'(f)f}{c(f)} - 1\right) \cdot \frac{f}{L^2c(f)} \cdot \mathcal{E}_f = 0 \tag{55}$$

$$1 + \mathcal{E}_x + \frac{c''(f)f}{c'(f)} \cdot \mathcal{E}_f = 0 \tag{56}$$

$$c(f)xN - 1 + c(f)xN \cdot \mathcal{E}_x + \mathcal{E}_N = 0 \tag{57}$$

Now remark that in equation (55) one has due to (48) and (12)

$$\frac{c'(f)f}{c(f)} - 1 = \mathcal{E}_c(f) - 1 = r_{\ln c}(f) - r_c(f) - 1 = -\frac{1}{1-r_u(x)} \tag{58}$$

while in equation (56) one has

$$\frac{c''(f)f}{c'(f)} = -r_c(f). \tag{59}$$

Moreover in equation (57) one has due to (54) and (52)

$$c(f)xN = 1 - \frac{f}{L} \cdot N = 1 - \frac{r_u(x)\,xc(f)}{1-r_u(x)} \cdot N$$

hence

$$\left(1 + \frac{r_u(x)}{1-r_u(x)}\right) c(f)xN = 1$$

i.e.

$$c(f)xN = 1 - r_u(x) \tag{60}$$

Thus, substituting (58) in (55), (59) in (56) and (60) in (57) we obtain the following three linear equations w.r.t. $\mathcal{E}_x$, $\mathcal{E}_f$ and $\mathcal{E}_N$:

$$(2 - r_{u'}(x)) \cdot \mathcal{E}_x - \mathcal{E}_f = r_u(x) - 1$$

$$\mathcal{E}_x - r_c(f) \cdot \mathcal{E}_f = -1$$

$$(1 - r_u(x)) \cdot \mathcal{E}_x + \mathcal{E}_N = r_u(x)$$

From these, one has

$$\mathcal{E}_f = (2 - r_{u'}(x)) \cdot \mathcal{E}_x - r_u(x) + 1 = (2 - r_{u'}(x)) \cdot (r_c(f) \cdot \mathcal{E}_f - 1) - r_u(x) + 1$$

i.e.

$$\mathcal{E}_f = \frac{1 + r_u(x) - r_{u'}(x)}{(2 - r_{u'}(x))\, r_c(f) - 1} = \frac{r'_u(x)\, x}{((2 - r_{u'}(x))\, r_c(f) - 1)\, r_u(x)}$$

hence formula (22) for $\mathcal{E}_f$ is proved.

Further, using (12), we can derive

$$\mathcal{E}_x = r_c(f) \cdot \mathcal{E}_f - 1 = \frac{(1 + r_u(x) - r_{u'}(x))\, r_c(f) - (2 - r_{u'}(x))\, r_c(f) + 1}{(2 - r_{u'}(x))\, r_c(f) - 1} =$$

$$= \frac{(1 - r_{\ln c}(f))\,(1 - r_u(x))}{(2 - r_{u'}(x))\, r_c(f) - 1}$$

hence formula (20) for $\mathcal{E}_x$ is proved.

Now let us calculate

$$\mathcal{E}_N = r_u(x) - (1 - r_u(x)) \cdot \mathcal{E}_x = r_u(x) - \frac{(1 - r_{\ln c}(f))\,(1 - r_u(x))^2}{(2 - r_{u'}(x))\, r_c(f) - 1} =$$

$$= 1 - \frac{(1 - r_u(x))\, r'_u(x)\, x r_c(f)}{(2 - r_{u'}(x))\, r_c(f) - 1}$$

hence formula (24) for $\mathcal{E}_N$ is proved.

Further,

$$\mathcal{E}_{Lx} = \mathcal{E}_x + 1 = \frac{(1 - r_{\ln c}(f))\,(1 - r_u(x))}{(2 - r_{u'}(x))\, r_c(f) - 1} + 1 = \frac{r'_u(x)\, x r_c(f)}{((2 - r_{u'}(x))\, r_c(f) - 1)\, r_u(x)}$$

hence formula (21) for $\mathcal{E}_{Lx}$ is proved.

Further, due to (12), one has, after some simplifications,

$$\mathcal{E}_{Nf} = \mathcal{E}_f + \mathcal{E}_N = \frac{r'_u(x) x}{((2 - r_{u'}(x))\, r_c(f) - 1)\, r_u(x)} + r_u(x) - \frac{(1 - r_{\ln c}(f))\,(1 - r_u(x))^2}{(2 - r_{u'}(x))\, r_c(f) - 1} =$$

$$= \frac{1 + r_u(x) - r_{u'}(x) - (1 - r_{\ln c}(f) + r_c(f))\,(1 - r_u(x))\,(1 - r_u(x)) + (1 - r_u(x))^2\, r_c(f)}{(2 - r_{u'}(x))\, r_c(f) - 1} + r_u(x) =$$

$$= \frac{(1 - r_{\ln c}(f))^2\,(1 - r_u(x))^2}{((2 - r_{u'}(x))\, r_c(f) - 1)\, r_c} + \frac{1}{r_c(f)} + r_u(x)$$

hence formula (23) for $\mathcal{E}_{Nf}$ is proved.

Further, one has from (15)

$$\frac{\partial p}{\partial L} = \left(\frac{c(f)}{1 - r_u(x)}\right)'_x \cdot \frac{\partial x}{\partial L} + \left(\frac{c(f)}{1 - r_u(x)}\right)'_f \cdot \frac{\partial f}{\partial L} = \frac{c(f) r'_u(x)}{(1 - r_u(x))^2} \cdot \frac{\partial x}{\partial L} + \frac{c'(f)}{1 - r_u(x)} \cdot \frac{\partial f}{\partial L}$$

hence

$$\frac{p}{L} \cdot \mathcal{E}_p = \frac{c(f) r_u'(x)}{(1 - r_u(x))^2} \cdot \frac{x}{L} \cdot \mathcal{E}_x + \frac{c'(f)}{1 - r_u(x)} \cdot \frac{f}{L} \cdot \mathcal{E}_f$$

therefore, due to (15) and (12),

$$\mathcal{E}_p = \frac{c(f) r_u'(x) x}{(1 - r_u(x))^2 p} \cdot \mathcal{E}_x + \frac{c'(f) f}{(1 - r_u(x)) p} \cdot \mathcal{E}_f =$$

$$= \frac{\left(1 - r_{\ln c}(f) + \frac{r_{\ln c}(f) - r_c(f)}{r_u(x)}\right) r_u'(x)\, x}{(2 - r_{u'}(x))\, r_c(f) - 1} = -\frac{r_c(f) r_u'(x)\, x}{(2 - r_{u'}(x))\, r_c(f) - 1}$$

hence formula (25) for $\mathcal{E}_p$ is proved.

As to elasticity of the markup, one has due to (50)

$$\mathcal{E}_{\frac{p-c}{p}/L} = \mathcal{E}_{r_u/L} = \mathcal{E}_{r_u/x} \cdot \mathcal{E}_{x/L} = \frac{r_u'(x)\, x}{r_u(x)} \cdot \mathcal{E}_{x/L} = \frac{r_u'(x)\, x}{r_u(x)} \cdot \frac{(1 - r_{\ln c}(f))\,(1 - r_u(x))}{(2 - r_{u'}(x))\, r_c(f) - 1}$$

hence formula (26) for $\mathcal{E}_{\frac{p-c}{p}/L}$ is proved.

Finally, the signs of elasticities, presented in the Table, can be obtained directly from the formulas for elasticities. The only thing to show is that

$$r_u'(x) < 0 \Longrightarrow r_{\ln c}(f) > 1 \tag{61}$$

Indeed, one has due to (13) and (12)

$$(1 + r_u(x) - r_{u'}(x) + 1 - r_u(x))\, r_c(f) > (1 - r_{\ln c}(f) + r_c(f))\,(1 - r_u(x))$$

i.e.

$$(1 + r_u(x) - r_{u'}(x))\, r_c(f) > (1 - r_{\ln c}(f))\,(1 - r_u(x))$$

$$r_u'(x) \cdot \frac{x r_c(f)}{r_u(x)} > (1 - r_{\ln c}(f))\,(1 - r_u(x))\,.$$

Since $r_c(f) > 0$, $r_u(x) > 0$ and moreover (see (13)) $1 - r_u(x) > 0$, implication(61) is shown.

12.3 Proof of Proposition 3

Using Appendix A, we apply Lemma 1, which predicts a monotone comparative statics of any extreme (minimal and maximal) roots $\hat{q}, \check{q}$ of any continuous equation $g(q, L) = 0$ w.r.t. parameter L, when function g changes the sign. (The intuition behind this lemma is simple: when we shift up any continuous curve whose left wing is above zero and the right wing is below zero—the roots must shift to the right, including the case of jumps). We apply this lemma to our main equilibrium equation $g(q, L) \equiv \mathcal{E}_R(q/L) - \mathcal{E}_{\mathbf{C}}(q) = 0$. We ensure the needed conditions $g(0, L) > 0$, $g(\infty, L) < 0$ by our general existence assumptions (18). Using our assumptions on u, c, both elasticities are continuous, so, existence of some roots $q_i : g(q, L) = 0$ is guaranteed. To state roots monotonicity like claim (i), we use $(\frac{\partial \mathcal{E}_R}{\partial L} > 0)$ in IED case (another case is proved similarly). Thereby comparing any two *single-valued* equilibria, we immediately get claim (i): $q_1 = \hat{q} < \check{q} = q_2$. However, the proof becomes more involved under multiple intersections of $\mathcal{E}_R(q/L), \mathcal{E}_{\mathbf{C}}(q)$, which means multi-valued argmaxima of profit. Which intersection relates to real equilibrium (which local argmaximum is global)? What happens at the points of jumps? Using our lemmas, we shall show that the "global" switch between the roots of FOC always goes in the same direction as the continuous changes in the local argmaxima. This argument uses the indirect way, through claim (iii) about behavior of λ.

Under non-trivial output $q^*(\lambda, L) > 0$, our profit $\pi(q, \lambda, L) \equiv q u'(q/L)/\lambda - C(q)$ decreases in λ and increases in L. Then (using Lemma 4), maximal profit $\pi^*(\lambda, L) \equiv \max_{q \geq 0} \pi(q, \lambda, L)$ is also a decreasing continuous function of λ and increasing continuous function of L. Therefore, (applying Lemma 1 to $g = \pi^*(\lambda, L)$) the free-entry equation $\pi^*(\lambda, L) = 0$ implies λ and L moving in the same direction, i.e., continuously *increasing* equilibrium root $\hat{\lambda}(L) > 0$ (which becomes indefinite only under trivial output $q^*(\lambda, L) = 0$).

More specifically, to derive the elasticity $\mathcal{E}_{\lambda(L)/L}$ of equilibrium λ, we totally differentiate w.r.t. L the free entry condition

$$\pi = q \cdot \frac{u'(q/L)}{\lambda(L)} - C(q) = 0$$

– ignoring q'_L (due to the envelope theorem). We get

$$\pi'_L = -q^2 \cdot \frac{u''(q/L)}{\lambda(L)L^2} - \frac{qu'(q/L)}{\lambda^2(L)} \cdot \lambda'(L) = 0 \Rightarrow$$

$$-q \cdot \frac{u''(q/L)}{u'(q/L)} = \frac{L^2}{\lambda(L)} \cdot \lambda'(L),$$

and obtain the needed elasticity, equal to that of the inverse demand:

$$\mathcal{E}_{\lambda(L)/L} = L \cdot \frac{\lambda'(L)}{\lambda(L)} = -\frac{q}{L} \cdot \frac{u''(q/L)}{u'(q/L)} = r_u(q/L).$$

Now, to detect single-crossing of profit, we can study the total derivative of marginal revenue R'_q at any q w.r.t. L (that must take into account the dependence $\lambda(L)$). Marginal revenue is

$$R'_q(q, L, \lambda(L)) \equiv \frac{u'(q/L) + qu''(q/L)/L}{\lambda(L)}.$$

At given q, the elasticity of the numerator w.r.t. L is

$$L \cdot \frac{\frac{d}{dL}[R'_q(q, L, \lambda(L))\lambda(L)]}{R'_q(q, L, \lambda(L))\lambda(L)} = L \cdot \frac{-2qu''(q/L)/L^2 - q^2 u'''(q/L)/L^3}{u'(q/L) + qu''(q/L)/L} =$$

$$= \frac{-2 - \frac{qu'''(q/L)}{Lu''(q/L)}}{\frac{u'(q/L)}{qu''(q/L)/L} + 1} = \frac{r_{u'}(q/L) - 2}{1 - \frac{1}{r_u(q/L)}},$$

whereas the elasticity of the denominator was just found as $r_u(q/L)$. Then, at given q, positive total elasticity of marginal revenue R'_q w.r.t. L means condition

$$\frac{r_{u'}(q/L) - 2}{1 - \frac{1}{r_u(q/L)}} - r_u(q/L) > 0 \Leftrightarrow \frac{r_{u'}(q/L) - 2}{1} < r_u(q/L)[1 - \frac{1}{r_u(q/L)}] \Leftrightarrow$$

$$\Leftrightarrow r_{u'}(q/L) - 1 < r_u(q/L).$$

This is a condition for supermodularity (strict single-crossing) of $\pi(q, L)$ along the equilibrium path accounting for $\lambda(L)$. We apply identity $\frac{r'_u(z) \cdot z}{r_u(z)} = 1 + r_u(z) - r_{u'}(z)$ to the above condition and reformulate it as

$$0 < 1 - r_{u'}(q/L) + r_u(q/L).$$

We conclude that $\mathcal{E}_{r_u} \equiv \frac{r'_u(x) \cdot x}{r_u(x)} > 0$ is a necessary and sufficient condition for supermodularity (strict single-crossing). Further, by Lemma 2, profit strict single-crossing in (x, L) is a necessary and sufficient condition for "strongly" non-decreasing output $q(L)$ along the equilibrium path, with or without jumps. Using Lemma 2 and IED condition ($r'_u > 0$) we conclude that all selections $q(L)$ non-decrease. Then any jumps (multiple argmaxima) are isolated points and must go downward. Taking into account this fact and that all singleton argmaxima increase, we get "strongly" *increasing* output $q(L)$ under IED (decreasing $q(L)$ under DED is proved similarly).

The outcomes for prices and masses of firms are evident from the equilibrium equations.

12.4 Proof of Proposition 4

One has

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{Lu(x)}{c(f)xL+f}\right) \equiv \frac{(c(f)xL+f)\,u'(x)-u(x)c(f)L}{(c(f)xL+f)^2}\cdot L = 0 \\ \frac{\partial}{\partial f} \equiv -\frac{(c'(f)xL+1)\,u(x)}{(c(f)xL+f)^2}\cdot L = 0 \end{cases}$$

Therefore, FOC is

$$\begin{cases} r_{\ln u} - r_u = \frac{cxL}{cxL+f} \\ c'xL = -1 \end{cases}$$

Further,

$$\frac{\partial^2}{\partial x^2}\left(\frac{Lu(x)}{c(f)xL+f}\right) \equiv \frac{\left((cxL+f)\,u'-ucL\right)'_x(cxL+f)-2\left((cxL+f)\,u'-ucL\right)(cxL+f)'_x}{(cxL+f)^3}\cdot L =$$

$$= \frac{cLu'+(cxL+f)\,u''-u'cL}{(cxL+f)^2}\cdot L = \frac{u''}{cxL+f}\cdot L = Nu'' < 0$$

$$\frac{\partial^2}{\partial f^2}\left(\frac{Lu(x)}{c(f)xL+f}\right) \equiv -\frac{\left((c'xL+1)\,u\right)'_f(cxL+f)-2\left(c'xL+1\right)'_f(c'xL+1)\,u(x)}{(cxL+f)^3}\cdot L =$$

$$= -\frac{c''xLu}{(cxL+f)^2}\cdot L = -c''xN^2u = -\frac{c''u'xN}{c} < 0$$

$$\frac{\partial^2}{\partial x\partial f}\left(\frac{Lu(x)}{c(f)xL+f}\right) \equiv \frac{\left((cxL+f)\,u'-ucL\right)'_f(cxL+f)-2\left(cxL+f\right)'_f\left((cxL+f)\,u'-ucL\right)}{(cxL+f)^3}\cdot L =$$

$$= \frac{(c'xL+1)\,u'-uc'L}{(cxL+f)^2}\cdot L = -\frac{uc'L^2}{(cxL+f)^2} = -uc'N^2 = -\frac{c'u'N}{c}$$

Hence

$$\det\begin{pmatrix} \left(\frac{Lu(x)}{c(f)xL+f}\right)''_{xx} & \left(\frac{Lu(x)}{c(f)xL+f}\right)''_{xf} \\ \left(\frac{Lu(x)}{c(f)xL+f}\right)''_{xf} & \left(\frac{Lu(x)}{c(f)xL+f}\right)''_{ff} \end{pmatrix} =$$

$$= -Nu''\cdot\frac{c''u'xN}{c} - \left(\frac{c'u'N}{c}\right)^2 = -\left(\frac{u'N}{f}\right)^2\mathcal{E}_c\cdot\left(\mathcal{E}_c + r_c r_u\right).$$

Therefore, SOC is

$$\mathcal{E}_c + r_c r_u > 0$$

12.5 Proof of Proposition 5

One has

$$0 = 1 - r_u - \frac{cxL}{cxL+f} = -r_u + \frac{f}{cxL+f}$$

Since $c'(f)xL = -1$, function f is a monotone function of x. Then

$$\frac{d}{dx}\left(1 - r_u - \frac{cxL}{cxL+f}\right) = -r'_u + \frac{(cxL+f)\,f'_x - (c'_x Lx + cL + f'_x)\,f}{(cxL+f)^2} =$$

$$= -r'_u + \frac{-fcL + \left(c - fc'_f\right)f'_x Lx}{(cxL+f)^2}$$

where

$$f'_x x = -\frac{c'_f}{c''_{ff}} > 0$$

(indeed, $c'(f)xL = -1 \Rightarrow c'_f + c''_{ff} x f'_x = 0$.) Hence

$$x \cdot \frac{d}{dx}\left(1 - r_u - \frac{cxL}{cxL+f}\right) = x \cdot \frac{d}{dx}\left(-r_u + \frac{f}{cxL+f}\right) = -r'_u x + \frac{-fcxL - \left(c - fc'_f\right)\frac{c'_f}{c''_{ff}}Lx}{(cxL+f)^2} =$$

$$= -\left(1 + r_u - r_{u'}\right) r_u - r_u \cdot \left(1 - r_u - \frac{1}{r_c}\right) = -\left(\left(2 - r_{u'}\right) r_c - 1\right) \cdot \frac{r_u}{r_c} < 0$$

Therefore, function $1 - r_u - \dfrac{cxL}{cxL+f}$, as function from x, decreases (with respect to x).

Recall:

Optimality: $r_{\ln u} - r_u - \dfrac{cxL}{cxL+f} = 0$

Equilibrium: $1 - r_u - \dfrac{cxL}{cxL+f} = 0$ **,** function $1 - r_u - \dfrac{cxL}{cxL+f}$ *decreases* with respect to x.

Moreover, $c'(f)Lx = -1 \Longrightarrow f'_x x = -\dfrac{c'_f}{c''_{ff}} > 0$, hence f *increases* with respect to x.

Moreover, $N = \dfrac{L}{cxL+f}$, hence N *decreases* with respect to x.

Therefore, the interconnection between optimal consumption (x^{opt}) and equilibrium consumption (x^*), optimal fixed costs (f^{opt}) and equilibrium fixed costs (f^*), optimal mass of firms (N^{opt}) and equilibrium mass of firms (N^*), are as in the Table.

Moreover, since

$$Nf = \frac{Lf}{cxL+f}$$

one has

$$(Nf)'_x = \frac{L}{(cxL+f)^2} \cdot \left((cxL+f)\,f'_x - \left(\left(c'_x x + c\right)L + f'_x\right)f\right) =$$

$$= -\frac{N^2 c c'_f}{c''_{ff}} \cdot \left(1 - \mathcal{E}_c - r_c\right) = -\frac{N^2 c c'_f}{c''_{ff}} \cdot \left(1 - r_{\ln c}\right).$$

Hence

$$r_{\ln c} < 1 \Longrightarrow (Nf)'_x > 0$$

$$r_{\ln c} = 1 \Longrightarrow (Nf)'_x = 0$$
$$r_{\ln c} > 1 \Longrightarrow (Nf)'_x < 0$$

Therefore, the interconnection between optimal total investments $(Nf)^{opt} = N^{opt} \cdot f^{opt}$ and equilibrium total investments $(Nf)^* = N^* \cdot f^*$ is as in the Table.

12.6 Proof of Proposition 6

One has

$$\begin{cases} r_{\ln u} - r_u = \dfrac{cxL}{cxL+f} \\ c'xL = -1 \\ N = \dfrac{L}{cxL+f} \end{cases}$$

Hence

$$\begin{cases} \left(r_{\ln u} - r_u - \dfrac{cxL}{cxL+f}\right)'_L + \left(r_{\ln u} - r_u - \dfrac{cxL}{cxL+f}\right)'_x \dfrac{\partial x}{\partial L} + \left(r_{\ln u} - r_u - \dfrac{cxL}{cxL+f}\right)'_f \dfrac{\partial f}{\partial L} = 0 \\ \left(c'xL\right)'_L + \left(c'xL\right)'_x \dfrac{\partial x}{\partial L} + \left(c'xL\right)'_f \dfrac{\partial f}{\partial L} = 0 \\ \left(N - \dfrac{L}{cxL+f}\right)'_L + \left(N - \dfrac{L}{cxL+f}\right)'_x \dfrac{\partial x}{\partial L} + \left(N - \dfrac{L}{cxL+f}\right)'_f \dfrac{\partial f}{\partial L} + \left(N - \dfrac{L}{cxL+f}\right)'_N \dfrac{\partial N}{\partial L} = 0 \end{cases}$$

i.e.

$$\begin{cases} \left(r'_{\ln u}x - r'_u x\right) \cdot \mathcal{E}_x + \left(1 - r_{\ln c}\right)\left(1 - \mathcal{E}_u\right)\mathcal{E}_u \cdot \mathcal{E}_f = 0 \\ 1 + \mathcal{E}_x - r_c \cdot \mathcal{E}_f = 0 \\ \mathcal{E}_u - 1 + \mathcal{E}_u \cdot \mathcal{E}_x + \mathcal{E}_N = 0 \end{cases}$$

hence

$$-\left(r'_{\ln u}x - r'_u x\right) + \left(\left(r'_{\ln u}x - r'_u x\right) r_c + \left(1 - r_{\ln c}\right)\left(1 - \mathcal{E}_u\right)\mathcal{E}_u\right) \cdot \mathcal{E}_f = 0$$

i.e.

$$\mathcal{E}_f = \frac{r'_{\ln u}x - r'_u x}{\left(r'_{\ln u}x - r'_u x\right) r_c + \left(1 - r_{\ln c}\right)\left(1 - \mathcal{E}_u\right)\mathcal{E}_u}$$

One has

$$r_{\ln u} - r_u = \frac{cxL}{cxL+f} = \frac{1}{1 - \mathcal{E}_c} = \frac{1}{1 - r_{\ln c} + r_c}$$

Hence

$$\left(1 - r_{\ln c} + r_c\right)\left(r_{\ln u} - r_u\right) = 1$$

Therefore

$$\left(r'_{\ln u}x - r'_u x\right) r_c + \left(1 - r_{\ln c}\right)\left(1 - \mathcal{E}_u\right)\mathcal{E}_u = -\left(\mathcal{E}_c + r_u r_c\right)\mathcal{E}_u < 0$$

Thus

$$\mathcal{E}_f = -\frac{\mathcal{E}'_u x}{\left(\mathcal{E}_c + r_u r_c\right)\mathcal{E}_u} = -\frac{1 - r_{\ln u}}{\mathcal{E}_c + r_u r_c}$$

$$\mathcal{E}_x = r_c \cdot \mathcal{E}_f - 1 = -\frac{(1-\mathcal{E}_u+\mathcal{E}_{u'})\,r_c}{\mathcal{E}_c + r_u r_c} - 1 = -\frac{(1-r_{\ln c})\,\mathcal{E}_c\mathcal{E}_u}{\mathcal{E}_c + r_u r_c}$$

$$\mathcal{E}_N = 1 - \mathcal{E}_u - \mathcal{E}_u \cdot \mathcal{E}_x = 1 + \frac{(1-r_{\ln u})\,r_c\mathcal{E}_u}{\mathcal{E}_c + r_u r_c} = \frac{\mathcal{E}_c + (1-\mathcal{E}_u)\,r_{\ln u} r_c}{\mathcal{E}_c + r_u r_c} =$$

(due to (29))

$$= \frac{\mathcal{E}_c - \mathcal{E}_c\mathcal{E}_u r_{\ln u} r_c}{\mathcal{E}_c + r_u r_c} = \left(\frac{(1-r_{\ln c})\,\mathcal{E}_u}{\mathcal{E}_c + r_u r_c} - 1\right)\mathcal{E}_u\mathcal{E}_c$$

Further, since $(1-\mathcal{E}_c)\,\mathcal{E}_u = 1$,

$$\mathcal{E}_{Lx} = 1 + \mathcal{E}_x = 1 - \frac{(1-r_{\ln c})\,\mathcal{E}_c\mathcal{E}_u}{\mathcal{E}_c + r_u r_c} = \frac{(r_{\ln u} - 1)\,r_c}{\mathcal{E}_c + r_u r_c}$$

$$\mathcal{E}_{Nf} = \mathcal{E}_f + \mathcal{E}_N = 1 - \frac{(1-r_{\ln u})\,(1-r_{\ln c})\,\mathcal{E}_u}{\mathcal{E}_c + r_u r_c}$$

Finally, the signs of elasticities, presented in the Table, can be obtained directly from the formulas for elasticities. The only thing to show is that

$$r_{\ln u} < 1 \Longrightarrow r_{\ln c} > 1. \tag{62}$$

Indeed, (29) means

$$\mathcal{E}_c = \frac{\mathcal{E}_u - 1}{\mathcal{E}_u}$$

Substitute this in (28), after simple calculations, we have

$$\frac{\mathcal{E}_u - 1}{\mathcal{E}_u} + r_u r_c > 0$$

i.e.

$$r_{\ln u} - 1 + (r_{\ln c} - 1)\,\mathcal{E}_u r_u > 0$$

hence (62) is shown.

12.7 Proof of Proposition 7

As usual, we use the following equations for equilibrium:

$$\frac{r_u(x)x}{1-r_u(x)} - \frac{f}{Lc(f,\alpha)} = 0 \tag{63}$$

$$c'_f(f,\alpha)Lx = -1 \tag{64}$$

(cf. (52) and (53)) and

$$p - \frac{c(f,\alpha)}{1-r_u(x)} = 0 \tag{65}$$

(cf. (15)). Thus

$$\left(\frac{r_u(x)x}{1-r_u(x)} - \frac{f}{Lc(f,\alpha)}\right)'_\alpha + \left(\frac{r_u(x)x}{1-r_u(x)} - \frac{f}{Lc(f,\alpha)}\right)'_x \cdot \frac{\partial x}{\partial \alpha} + \left(\frac{r_u(x)x}{1-r_u(x)} - \frac{f}{Lc(f,\alpha)}\right)'_f \cdot \frac{\partial f}{\partial \alpha} = 0$$

$$\left(c'_f(f,\alpha)Lx\right)'_\alpha + \left(c'_f(f,\alpha)Lx\right)'_x \cdot \frac{\partial x}{\partial \alpha} + \left(c'_f(f,\alpha)Lx\right)'_f \cdot \frac{\partial f}{\partial \alpha} = 0$$

$$\left(p-\frac{c(f,\alpha)}{1-r_u(x)}\right)'_\alpha+\left(p-\frac{c(f,\alpha)}{1-r_u(x)}\right)'_p\cdot\frac{\partial p}{\partial\alpha}+\left(p-\frac{c(f,\alpha)}{1-r_u(x)}\right)'_x\cdot\frac{\partial x}{\partial\alpha}+\left(p-\frac{c(f,\alpha)}{1-r_u(x)}\right)'_f\cdot\frac{\partial f}{\partial\alpha}=0$$

i.e.

$$\frac{c'_\alpha(f,\alpha)f}{Lc^2(f,\alpha)}+\frac{r'_u(x)x+(1-r_u(x))\,r_u(x)}{(1-r_u(x))^2}\cdot\frac{\partial x}{\partial\alpha}-\frac{c(f,\alpha)-c'_f(f,\alpha)f}{Lc^2(f,\alpha)}\cdot\frac{\partial f}{\partial\alpha}=0$$

$$c''_{f\alpha}(f,\alpha)Lx+c'_f(f,\alpha)L\cdot\frac{\partial x}{\partial\alpha}+c''_{ff}(f,\alpha)Lx\cdot\frac{\partial f}{\partial\alpha}=0$$

$$-\frac{c'_\alpha(f,\alpha)}{1-r_u(x)}+\frac{\partial p}{\partial\alpha}-\frac{r'_u(x)c(f,\alpha)}{(1-r_u(x))^2}\cdot\frac{\partial x}{\partial\alpha}-\frac{c'_f(f,\alpha)}{1-r_u(x)}\cdot\frac{\partial f}{\partial\alpha}=0$$

i.e.

$$\frac{c'_\alpha(f,\alpha)f}{Lc^2(f,\alpha)}+\frac{(2-r_{u'}(x))\,r_u(x)}{(1-r_u(x))^2}\cdot\frac{x}{\alpha}\cdot\mathcal{E}_{x/\alpha}-\frac{c(f,\alpha)-c'_f(f,\alpha)f}{Lc^2(f,\alpha)}\cdot\frac{f}{\alpha}\cdot\mathcal{E}_{f/\alpha}=0$$

$$c''_{f\alpha}(f,\alpha)Lx+c'_f(f,\alpha)L\cdot\frac{x}{\alpha}\cdot\mathcal{E}_{x/\alpha}+c''_{ff}(f,\alpha)Lx\cdot\frac{f}{\alpha}\cdot\mathcal{E}_{f/\alpha}=0$$

$$-\frac{c'_\alpha(f,\alpha)}{1-r_u(x)}+\frac{p}{\alpha}\cdot\mathcal{E}_{p/\alpha}-\frac{r'_u(x)c(f,\alpha)}{(1-r_u(x))^2}\cdot\frac{x}{\alpha}\cdot\mathcal{E}_{x/\alpha}-\frac{c'_f(f,\alpha)}{1-r_u(x)}\cdot\frac{f}{\alpha}\cdot\mathcal{E}_{f/\alpha}=0$$

i.e. (due to (63) and (65))

$$\frac{c'_\alpha(f,\alpha)f\alpha}{Lc^2(f,\alpha)}+\frac{(2-r_{u'}(x))}{1-r_u(x)}\cdot\frac{f}{Lc(f,\alpha)}\cdot\mathcal{E}_{x/\alpha}-\frac{c(f,\alpha)-c'_f(f,\alpha)f}{Lc^2(f,\alpha)}\cdot f\cdot\mathcal{E}_{f/\alpha}=0$$

$$\frac{c''_{f\alpha}(f,\alpha)\alpha}{c'_f(f,\alpha)}+\mathcal{E}_{x/\alpha}+\frac{c''_{ff}(f,\alpha)f}{c'_f(f,\alpha)}\cdot\mathcal{E}_{f/\alpha}=0$$

$$-\frac{c'_\alpha(f,\alpha)\alpha}{1-r_u(x)}+\frac{c(f,\alpha)}{1-r_u(x)}\cdot\mathcal{E}_{p/\alpha}-\frac{r'_u(x)xc(f,\alpha)}{(1-r_u(x))^2}\cdot\mathcal{E}_{x/\alpha}-\frac{c'_f(f,\alpha)f}{1-r_u(x)}\cdot\mathcal{E}_{f/\alpha}=0$$

i.e.

$$\mathcal{E}_{c/\alpha}+\frac{(2-r_{u'}(x))}{1-r_u(x)}\cdot\mathcal{E}_{x/\alpha}-\left(1-\mathcal{E}_{c/f}\right)\cdot\mathcal{E}_{f/\alpha}=0$$

$$\mathcal{E}_{c'_f/\alpha}+\mathcal{E}_{x/\alpha}-r_c(f,\alpha)\cdot\mathcal{E}_{f/\alpha}=0$$

$$-\mathcal{E}_{c/\alpha}+\mathcal{E}_{p/\alpha}-\frac{r'_u(x)x}{1-r_u(x)}\cdot\mathcal{E}_{x/\alpha}-\mathcal{E}_{c/f}\cdot\mathcal{E}_{f/\alpha}=0$$

i.e. (due to (12))

$$(1-r_u(x))\cdot\mathcal{E}_{c/\alpha}+(2-r_{u'}(x))\cdot\mathcal{E}_{x/\alpha}-\mathcal{E}_{f/\alpha}=0 \tag{66}$$

$$\mathcal{E}_{x/\alpha}=r_c(f,\alpha)\cdot\mathcal{E}_{f/\alpha}-\mathcal{E}_{c'_f/\alpha} \tag{67}$$

$$\mathcal{E}_{p/\alpha}=\frac{r'_u(x)x}{1-r_u(x)}\cdot\mathcal{E}_{x/\alpha}+\mathcal{E}_{c/f}\cdot\mathcal{E}_{f/\alpha}+\mathcal{E}_{c/\alpha} \tag{68}$$

hence (from (67) and (66))

$$\mathcal{E}_{x/\alpha}=r_c\cdot\left((1-r_u(x))\cdot\mathcal{E}_{c/\alpha}+(2-r_{u'}(x))\cdot\mathcal{E}_{x/\alpha}\right)-\mathcal{E}_{c'_f/\alpha}$$

$$\mathcal{E}_{x/\alpha}=\frac{-(1-r_u(x))\,r_c\mathcal{E}_{c/\alpha}+\mathcal{E}_{c'_f/\alpha}}{(2-r_{u'}(x))\,r_c-1}>0 \tag{69}$$

and (from (67), using (69))

$$\mathcal{E}_{f/\alpha} = \frac{\mathcal{E}_{x/\alpha} + \mathcal{E}_{c'_f/\alpha}}{r_c} = \frac{-\left(1 - r_u(x)\right) r_c \mathcal{E}_{c/\alpha} + \mathcal{E}_{c'_f/\alpha} + \left(\left(2 - r_{u'}(x)\right) r_c - 1\right) \mathcal{E}_{c'_f/\alpha}}{\left(\left(2 - r_{u'}(x)\right) r_c - 1\right) r_c}$$

$$\mathcal{E}_{f/\alpha} = \frac{-\left(1 - r_u(x)\right) \mathcal{E}_{c/\alpha} + \left(2 - r_{u'}(x)\right) \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} > 0 \tag{70}$$

and (from (68), using (69) and (70))

$$\mathcal{E}_{p/\alpha} = \frac{r'_u(x)x}{1 - r_u(x)} \cdot \frac{-\left(1 - r_u(x)\right) r_c \mathcal{E}_{c/\alpha} + \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} + \mathcal{E}_{c/f} \cdot \frac{-\left(1 - r_u(x)\right) \mathcal{E}_{c/\alpha} + \left(2 - r_{u'}(x)\right) \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} + \mathcal{E}_{c/\alpha} =$$

$$= \frac{-r'_u(x)xr_c\mathcal{E}_{c/\alpha} + \dfrac{r'_u(x)x}{1 - r_u(x)} \cdot \mathcal{E}_{c'_f/\alpha} - \left(1 - r_u(x)\right) \mathcal{E}_{c/f}\mathcal{E}_{c/\alpha} + \left(2 - r_{u'}(x)\right) \mathcal{E}_{c/f}\mathcal{E}_{c'_f/\alpha} + \left(\left(2 - r_{u'}(x)\right) r_c - 1\right) \mathcal{E}_{c/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} =$$

$$= \frac{\left(-r'_u(x)xr_c - \left(1 - r_u(x)\right) \mathcal{E}_{c/f} + \left(2 - r_{u'}(x)\right) r_c - 1\right) \mathcal{E}_{c/\alpha} + \left(\dfrac{r'_u(x)x}{1 - r_u(x)} + \left(2 - r_{u'}(x)\right) \mathcal{E}_{c/f}\right) \cdot \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} =$$

(since $r'_u(x)x = \left(1 + r_u(x) - r_{u'}(x)\right) r_u(x)$ and, due to (63) and (64), $\mathcal{E}_{c/f} = \dfrac{r_u(x)}{r_u(x) - 1}$)

$$= \frac{\left(-\left(1 + r_u - r_{u'}\right) r_u r_c + r_u + \left(2 - r_{u'}\right) r_c - 1\right) \mathcal{E}_{c/\alpha} + \left(\dfrac{\left(1 + r_u - r_{u'}\right) r_u}{1 - r_u} + \left(2 - r_{u'}\right) \cdot \dfrac{r_u}{r_u - 1}\right) \cdot \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}\right) r_c - 1} =$$

$$= \frac{\left(\left(-\left(2 - 1 + r_u - r_{u'}\right) r_u + 2 - r_{u'}\right) r_c - \left(1 - r_u\right)\right) \mathcal{E}_{c/\alpha} + \left(1 + r_u - r_{u'} - 2 + r_{u'}\right) \cdot \dfrac{r_u}{1 - r_u} \cdot \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}\right) r_c - 1} =$$

$$= \frac{\left(\left(2 - r_{u'} + r_u\right) r_c - 1\right) \left(1 - r_u\right) \mathcal{E}_{c/\alpha} - r_u \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}\right) r_c - 1} =$$

$$= \frac{\left(\left(2 - r_{u'}\right) r_c - 1 + r_u r_c\right) \left(1 - r_u\right) \mathcal{E}_{c/\alpha} - r_u \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}\right) r_c - 1} < 0$$

since $\left(2 - r_{u'}\right) r_c - 1 > 0$, $0 < r_u < 1$, $r_c > 0$, $\mathcal{E}_{c/\alpha} < 0$ and $\mathcal{E}_{c'_f/\alpha} > 0$.

Further, (see (16))

$$\mathcal{E}_{Nf/\alpha} = \mathcal{E}_{\frac{Nf}{L}/\alpha} = \mathcal{E}_{r_u(x)/\alpha} = \mathcal{E}_{r_u(x)/x} \cdot \mathcal{E}_{x/\alpha} = \frac{r'_u(x)x}{r_u(x)} \cdot \mathcal{E}_{x/\alpha}$$

and (using (69) and (70))

$$\mathcal{E}_{N/\alpha} = \mathcal{E}_{Nf/\alpha} - \mathcal{E}_{f/\alpha} = \frac{r'_u(x)x}{r_u(x)} \cdot \mathcal{E}_{x/\alpha} - \mathcal{E}_{f/\alpha} =$$

$$= \frac{r'_u(x)x}{r_u(x)} \cdot \frac{-\left(1 - r_u(x)\right) r_c \mathcal{E}_{c/\alpha} + \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} - \frac{-\left(1 - r_u(x)\right) \mathcal{E}_{c/\alpha} + \left(2 - r_{u'}(x)\right) \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} =$$

$$= \frac{\left(1 + r_u(x) - r_{u'}(x)\right) \left(-\left(1 - r_u(x)\right) r_c \mathcal{E}_{c/\alpha} + \mathcal{E}_{c'_f/\alpha}\right) + \left(1 - r_u(x)\right) \mathcal{E}_{c/\alpha} - \left(2 - r_{u'}(x)\right) \mathcal{E}_{c'_f/\alpha}}{\left(2 - r_{u'}(x)\right) r_c - 1} =$$

$$= \frac{\left(-\left(1+r_u(x)-r_{u'}(x)\right)r_c+1\right)\left(1-r_u(x)\right)\mathcal{E}_{c/\alpha}+\left(1+r_u(x)-r_{u'}(x)-2+r_{u'}(x)\right)\mathcal{E}_{c'_f/\alpha}}{\left(2-r_{u'}(x)\right)r_c-1} =$$

$$= \frac{\left(-\left(1+r_u(x)-r_{u'}(x)\right)r_c+1\right)\left(1-r_u(x)\right)\mathcal{E}_{c/\alpha}-\left(1-r_u(x)\right)\mathcal{E}_{c'_f/\alpha}}{\left(2-r_{u'}(x)\right)r_c-1} =$$

$$= \frac{\left(\left(\left(1-r_u(x)\right)r_c-\left(2-r_{u'}(x)\right)r_c+1\right)\mathcal{E}_{c/\alpha}-\mathcal{E}_{c'_f/\alpha}\right)\left(1-r_u(x)\right)}{\left(2-r_{u'}(x)\right)r_c-1}$$

Finally, the signs of elasticities, presented in the Table, can be obtained directly from the formulas for elasticities.

12.8 Proof of Proposition 8

System of equilibrium equations:

$$\frac{r_u\left(x,\beta\right)x}{1-r_u\left(x,\beta\right)}-\frac{f}{Lc(f)}=0 \tag{71}$$

$$c'(f)Lx=-1 \tag{72}$$

$$\left(c(f)xL+f\right)N=L \tag{73}$$

Let us calculate the total derivatives w.r.t. β:

$$\left(\frac{r_u\left(x,\beta\right)x}{1-r_u\left(x,\beta\right)}-\frac{f}{Lc(f)}\right)'_\beta+\left(\frac{r_u\left(x,\beta\right)x}{1-r_u\left(x,\beta\right)}-\frac{f}{Lc(f)}\right)'_x\cdot\frac{\partial x}{\partial\beta}+\left(\frac{r_u\left(x,\beta\right)x}{1-r_u\left(x,\beta\right)}-\frac{f}{Lc(f)}\right)'_f\cdot\frac{\partial f}{\partial\beta}=0$$

$$\left(c'(f)Lx\right)'_\beta+\left(c'(f)Lx\right)'_x\cdot\frac{\partial x}{\partial\beta}+\left(c'(f)Lx\right)'_f\cdot\frac{\partial f}{\partial\beta}=0$$

$$\left(\left(c(f)xL+f\right)N\right)'_\beta+\left(\left(c(f)xL+f\right)N\right)'_x\cdot\frac{\partial x}{\partial\beta}+\left(\left(c(f)xL+f\right)N\right)'_f\cdot\frac{\partial f}{\partial\beta}+\left(\left(c(f)xL+f\right)N\right)'_N\cdot\frac{\partial N}{\partial\beta}=0$$

i.e.

$$\left(\frac{r_u\left(x,\beta\right)x}{1-r_u\left(x,\beta\right)}\right)'_\beta+\left(\frac{r_u\left(x,\beta\right)x}{1-r_u\left(x,\beta\right)}\right)'_x\cdot\frac{\partial x}{\partial\beta}+\left(-\frac{f}{Lc(f)}\right)'_f\cdot\frac{\partial f}{\partial\beta}=0$$

$$c'(f)L\cdot\frac{\partial x}{\partial\beta}+c''(f)Lx\cdot\frac{\partial f}{\partial\beta}=0$$

$$c(f)LN\cdot\frac{\partial x}{\partial\beta}+\left(c'(f)xL+1\right)N\cdot\frac{\partial f}{\partial\beta}+\left(c(f)xL+f\right)\cdot\frac{\partial N}{\partial\beta}=0$$

i.e., using (71), (72), (73), etc.,

$$\frac{\frac{\partial r_u\left(x,\beta\right)}{\partial\beta}\cdot x}{\left(1-r_u\left(x,\beta\right)\right)^2}+\frac{\left(2-r_{u'}\left(x,\beta\right)\right)\cdot r_u\left(x,\beta\right)}{\left(1-r_u\left(x,\beta\right)\right)^2}\cdot\frac{\partial x}{\partial\beta}-\frac{c(f)-fc'(f)}{L\left(c(f)\right)^2}\cdot\frac{\partial f}{\partial\beta}=0$$

$$c'(f)L\cdot\frac{\partial x}{\partial\beta}+c''(f)Lx\cdot\frac{\partial f}{\partial\beta}=0$$

$$c(f)LN\cdot\frac{\partial x}{\partial\beta}+\left(c(f)xL+f\right)\cdot\frac{\partial N}{\partial\beta}=0$$

i.e.

$$\frac{\frac{\partial r_u(x,\beta)}{\partial \beta}\cdot x}{(1-r_u(x,\beta))^2}+\frac{(2-r_{u'}(x,\beta))\cdot r_u(x,\beta)}{(1-r_u(x,\beta))^2}\cdot\frac{x}{\beta}\cdot\mathcal{E}_{x/\beta}-\frac{c(f)-fc'(f)}{L(c(f))^2}\cdot\frac{f}{\beta}\cdot\mathcal{E}_{f/\beta}=0$$

$$c'(f)L\cdot\frac{x}{\beta}\cdot\mathcal{E}_{x/\beta}+c''(f)Lx\cdot\frac{f}{\beta}\cdot\mathcal{E}_{f/\beta}=0$$

$$c(f)LN\cdot\frac{x}{\beta}\cdot\mathcal{E}_{x/\beta}+(c(f)xL+f)\cdot\frac{N}{\beta}\cdot\mathcal{E}_{N/\beta}=0$$

i.e.

$$\frac{\frac{\partial r_u(x,\beta)}{\partial \beta}\cdot \beta}{(1-r_u(x,\beta))^2}+\frac{(2-r_{u'}(x,\beta))\cdot r_u(x,\beta)}{(1-r_u(x,\beta))^2}\cdot\mathcal{E}_{x/\beta}-(1-\mathcal{E}_c(f))\cdot\frac{f}{c(f)xL}\cdot\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}-r_c(f)\cdot\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}+\frac{c(f)xL+f}{c(f)Lx}\cdot\mathcal{E}_{N/\beta}=0$$

i.e.

$$\mathcal{E}_{r_u/\beta}(x,\beta)\cdot r_u(x,\beta)+(2-r_{u'}(x,\beta))\cdot r_u(x,\beta)\cdot\mathcal{E}_{x/\beta}-(1-r_u(x,\beta))\cdot\frac{f}{c(f)xL}\cdot\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}-r_c(f)\cdot\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}+(1-\mathcal{E}_c(f))\cdot\mathcal{E}_{N/\beta}=0$$

i.e.

$$\mathcal{E}_{r_u/\beta}(x,\beta)+(2-r_{u'}(x,\beta))\cdot\mathcal{E}_{x/\beta}-\frac{1-r_u(x,\beta)}{r_u(x,\beta)}\cdot\frac{f}{c(f)xL}\cdot\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}-r_c(f)\cdot\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}+(1-\mathcal{E}_c(f))\cdot\mathcal{E}_{N/\beta}=0$$

i.e.

$$\mathcal{E}_{r_u/\beta}(x,\beta)+(2-r_{u'}(x,\beta))\cdot\mathcal{E}_{x/\beta}-\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}-r_c(f)\cdot\mathcal{E}_{f/\beta}=0$$

$$\mathcal{E}_{x/\beta}+(1-\mathcal{E}_c(f))\cdot\mathcal{E}_{N/\beta}=0$$

i.e.

$$\mathcal{E}_{f/\beta}=-\frac{\mathcal{E}_{r_u/\beta}(x,\beta)}{(2-r_{u'}(x,\beta))\cdot r_c(f)-1}<0$$

$$\mathcal{E}_{q/\beta}=\mathcal{E}_{Lx/\beta}=\mathcal{E}_{x/\beta}=r_c(f)\cdot\mathcal{E}_{f/\beta}<0$$

$$\mathcal{E}_{N/\beta}=-\frac{\mathcal{E}_{x/\beta}}{1-\mathcal{E}_c(f)}=-\frac{r_c(f)}{1-\mathcal{E}_c(f)}\cdot\mathcal{E}_{f/\beta}>0$$

Moreover,

$$\mathcal{E}_{\frac{p-c}{p}/\beta}=\mathcal{E}_{\frac{Nf}{L}/\beta}=\mathcal{E}_{Nf/\beta}=\mathcal{E}_{N/\beta}+\mathcal{E}_{f/\beta}=\left(1-\frac{r_c(f)}{1-\mathcal{E}_c(f)}\right)\cdot\mathcal{E}_{f/\beta}=\frac{1-\mathcal{E}_c(f)-r_c(f)}{1-\mathcal{E}_c(f)}\cdot\mathcal{E}_{f/\beta}=\frac{1-r_{\ln c}(f)}{1-\mathcal{E}_c(f)}\cdot\mathcal{E}_{f/\beta}$$

Finally,

$$\frac{\partial p}{\partial\beta}=\left(\frac{c(f)}{1-r_u(x,\beta)}\right)'_\beta+\left(\frac{c(f)}{1-r_u(x,\beta)}\right)'_x\cdot\frac{\partial x}{\partial\beta}+\left(\frac{c(f)}{1-r_u(x,\beta)}\right)'_f\cdot\frac{\partial f}{\partial\beta}=$$

$$=\frac{\frac{\partial r_u(x,\beta)}{\partial\beta}\cdot c(f)}{(1-r_u(x,\beta))^2}+\frac{r_u'(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2}\cdot\frac{x}{\beta}\cdot\mathcal{E}_{x/\beta}+\frac{c'(f)}{1-r_u(x,\beta)}\cdot\frac{f}{\beta}\cdot\mathcal{E}_{f/\beta}=$$

$$=\left(\frac{\frac{\partial r_u(x,\beta)}{\partial\beta}\cdot\beta}{r_u(x,\beta)}+\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot\mathcal{E}_{x/\beta}+\frac{1-r_u(x,\beta)}{r_u(x,\beta)}\cdot\frac{c'(f)f}{c(f)}\cdot\mathcal{E}_{f/\beta}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\mathcal{E}_{r_u/\beta}(x,\beta)+\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot\mathcal{E}_{x/\beta}+\frac{1-r_u(x,\beta)}{r_u(x,\beta)}\cdot\mathcal{E}_c\cdot\mathcal{E}_{f/\beta}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\mathcal{E}_{r_u/\beta}(x,\beta)+\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot r_c(f)\cdot\mathcal{E}_{f/\beta}+\frac{1-r_u(x,\beta)}{r_u(x,\beta)}\cdot\mathcal{E}_c\cdot\mathcal{E}_{f/\beta}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\mathcal{E}_{r_u/\beta}(x,\beta)+\left(\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot r_c(f)+\frac{1-r_u(x,\beta)}{r_u(x,\beta)}\cdot\mathcal{E}_c\right)\cdot\mathcal{E}_{f/\beta}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\mathcal{E}_{r_u/\beta}(x,\beta)+\left(\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot r_c(f)+\frac{1-r_u(x,\beta)}{r_u(x,\beta)}\cdot\mathcal{E}_c\right)\cdot\mathcal{E}_{f/\beta}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\mathcal{E}_{r_u/\beta}(x,\beta)+\left(\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot r_c(f)-\frac{c(f)}{c'(f)f}\cdot\mathcal{E}_c\right)\cdot\mathcal{E}_{f/\beta}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\mathcal{E}_{r_u/\beta}(x,\beta)+\left(\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot r_c(f)-1\right)\cdot\mathcal{E}_{f/\beta}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\mathcal{E}_{r_u/\beta}(x,\beta)-\left(\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot r_c(f)-1\right)\cdot\frac{\mathcal{E}_{r_u/\beta}(x,\beta)}{(2-r_{u'}(x,\beta))\cdot r_c(f)-1}\right)\cdot\frac{r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(1-\frac{\frac{r_u'(x,\beta)\,x}{r_u(x,\beta)}\cdot r_c(f)-1}{(2-r_{u'}(x,\beta))\cdot r_c(f)-1}\right)\cdot\frac{\mathcal{E}_{r_u/\beta}(x,\beta)\,r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(1-\frac{(1+r_u(x,\beta)-r_{u'}(x,\beta))\cdot r_c(f)-1}{(2-r_{u'}(x,\beta))\cdot r_c(f)-1}\right)\cdot\frac{\mathcal{E}_{r_u/\beta}(x,\beta)\,r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$=\left(\frac{(2-r_{u'}(x,\beta))\cdot r_c(f)-(1+r_u(x,\beta)-r_{u'}(x,\beta))\cdot r_c(f)}{(2-r_{u'}(x,\beta))\cdot r_c(f)-1}\right)\cdot\frac{\mathcal{E}_{r_u/\beta}(x,\beta)\,r_u(x,\beta)\,c(f)}{(1-r_u(x,\beta))^2\,\beta}=$$

$$= \frac{\mathcal{E}_{r_u/\beta}\left(x,\beta\right) r_u\left(x,\beta\right) c(f) r_c\left(f\right)}{\left(\left(2-r_{u'}\left(x,\beta\right)\right)\cdot r_c\left(f\right)-1\right)\left(1-r_u\left(x,\beta\right)\right)\beta} = -\frac{\mathcal{E}_{r_u/\beta}\left(x,\beta\right) r_c\left(f\right) f c'(f)}{\left(\left(2-r_{u'}\left(x,\beta\right)\right)\cdot r_c\left(f\right)-1\right)\beta}\,.$$

Hence

$$\mathcal{E}_{p/\beta} = \frac{\partial p}{\partial \beta}\cdot\frac{\beta}{p} = -\frac{\mathcal{E}_{r_u/\beta}\left(x,\beta\right) r_c\left(f\right) f c'(f)}{\left(\left(2-r_{u'}\left(x,\beta\right)\right)\cdot r_c\left(f\right)-1\right)}\cdot\frac{1-r_u\left(x,\beta\right)}{c(f)} = -\frac{\left(1-r_u\left(x,\beta\right)\right)\mathcal{E}_{r_u/\beta}\left(x,\beta\right) r_c\left(f\right)\mathcal{E}_c\left(f\right)}{\left(2-r_{u'}\left(x,\beta\right)\right)\cdot r_c\left(f\right)-1} > 0.$$

13 Appendix C: The Table of the notations

In this Section we present the Table of all the notations and their definitions.

notation	definition
L	the (big) number of identical consumers/workers (market size)
$[0, N]$	an endogenous interval of identical firms producing varieties of some "differentiated good"
N	the endogenous mass of firms, or the scope (the interval) of varieties
x_i	consumption of i-th variety by any consumer, $i \in [0, N]$
X	$X = (x_i)_{i \leq N}$, $X : [0, N] \to \mathbb{R}_+$, an infinite-dimensional consumption vector
p_i	the price of $i - th$ variety by any consumer, $i \in [0, N]$
x	consumption of any variety by any consumer, in symmetric case ($x_i = x \, \forall i \in [0, N]$)
p	the price of any variety by any consumer, in symmetric case ($p_i = p \, \forall i \in [0, N]$)
$\mathcal{E}_g(z)$	$\mathcal{E}_g(z) \equiv \dfrac{zg'(z)}{g(z)}$, the elasticity
$r_g(z)$	$r_g(z) \equiv -\dfrac{zg''(z)}{g'(z)}$, the Arrow-Pratt measure of concavity, $r_g(z) \equiv -\mathcal{E}_{g'}(z)$
$u(\cdot)$	the Bernoulli utility function, $u'(x) > 0,\ u''(x) < 0$
RLV	"relative love for variety", $r_u(x) \equiv -\dfrac{xu''(x)}{u'(x)}$
λ	the Lagrange multiplier of the budget constraint (the marginal utility of income)
$R\&D$	the Research and Development activities
q	$q = L \cdot x$ producing of a firm (firm size)
f	fixed costs of a firm (investments in R&D)
$c\,(f)$	marginal costs, "innovation function", $c'(f) < 0$
FOC	First Order Condition
SOC	Second Order Condition
IED	Increasing Elasticity of Demand (the case $r'_u\,(x) > 0$)
DED	Decreasing Elasticity of Demand (the case $r'_u\,(x) < 0$)
DEU	Decreasing Elasticity of Utility (the case $\mathcal{E}'_u\,(x) < 0$)
IEU	Increasing Elasticity of Utility (the case $\mathcal{E}'_u\,(x) > 0$)
CES	Constant Elasticity of Substitution ($u\,(x)$ is CES-function $\Longleftrightarrow r'_u\,(x) = \mathcal{E}'_u\,(x) = 0$)
α	in marginal costs $c = c(f, \alpha)$, a technological innovation parameter, $\frac{\partial c}{\partial \alpha} < 0,\ \frac{\partial^2 c}{\partial f \partial \alpha} < 0$
β	in parameterized utility $u = u\,(x, \beta)$, a parameter of inter-industry comparisons, $\frac{\partial r_u(x,\beta)}{\partial \beta} > 0$
$\gamma(i)$	continuous density of distribution of heterogeneous abilities, defined on $[0, \infty)$
$\Gamma(t)$	$\Gamma(t) = \int_0^t \gamma(i) di$ is the cumulative probability

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