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ENDOGENOUS STRUCTURE OF CITIES: TRADE, COMMUTING, COMMUNICATION

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Endogenous Structure of Cities: Trade, Commuting, Communication*

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Abstract

The purpose of paper is to investigate how the interplay of trade, commuting and communication costs shapes economy at both inter-regional and intra-urban level. Specifically, we study how trade affects the internal structure of cities and how decentralizing the production and consumption of goods in secondary employment centers allows firms located in a large city to maintain their predominance. The feature of approach is using of two-dimensional city pattern instead of the "long narrow city" model.

Keywords and Phrases: city structure; secondary business center; commuting cost; trade cost; communication cost

JEL Classification Numbers: F12, F22, R12, R14

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1 Introduction

Spatial economics has acquired new life since publication of Krugman's (1991) pioneering paper. Combined increasing returns, imperfect competition, commodity trade and the mobility of production factors Krugman has formed his now famous "core-periphery" model. Such a combination contradicts to the mainstream paradigm of constant returns and perfect competition, which has dominated in economic theory for a long time. Furthermore, to the trade-off between increasing returns and transport costs Krugman (1980) has added a third factor: the size of spatially separated markets. The main achievement of New Economic Geography (NEG) was to show *how market size interacts with scale economies internal to firms and transport costs to shape the space-economy*.

In NEG, the market outcome arises from the interplay between a dispersion force and an agglomeration force operating within a general equilibrium model. In Krugman (1991) and Fujita et al. (1999), the dispersion force ensures from the spatial immobility of farmers. As for the agglomeration force, Krugman (1991, p.486) noticed that *circular causation* a la Myrdal (1957) takes place because the following two effects reinforce each other: "manufactures production will tend to concentrate where there is a large market, but the market will be large where manufactures production is concentrated."

In this framework, however, the internal structure of regions was not accounted for. In the present paper we consider NEG models which allows for the internal structure of urban agglomerations through the introduction of a land market. To be precise, we start by focusing on the causes and consequences of the internal structure of cities, because the way they are organized has a major impact of the well-being of people. In particular, housing and commuting costs, which we call *urban costs*, account for a large share of consumers' expenditures. At this point we are agree with Helpman (1998) for whom *urban costs are the main dispersion force at work* in modern urbanized economies. In our setting, an agglomeration is structured as a monocentric city in which firms gather in a central business district. Competition for land among consumers gives rise to land rent and commuting costs that both increase with population size. In other words, our approach endows regions with an urban structure which is absent in standard NEG models.

As a result, the space-economy is the outcome of the interaction between two types of mobility costs: the transport costs of commodities and the commuting costs borne by workers. Evolution of commuting costs within cities, instead of transport costs between cities, becomes the key-factor explaining how the space-economy is organized. Moreover, despite the many advantages provided by the inner city through an easy access to highly specialized services, the significant fall in communication costs has led firms or developers to form enterprise zones or edge cities (Henderson and Mitra 1996). We then go one step further by allowing firms to form secondary business centers. This analysis shows how polycentricity alleviates the urban of urban costs, which allows a big city to retain its dominant position by accommodating a large share of activities.

Creation of subcenters within a city, i.e. the formation of a polycentric city, appears to be a natural way to alleviate the burden of urban costs. It is, therefore, no surprise that Anas et al. (1998) observe that "polycentricity is an increasingly prominent feature of the landscape." Thus, the escalation of urban costs in large cities seems to prompt a redeployment of activities in a polycentric pattern, while smaller cities retain their monocentric shape. However, for this to happen, firms set up in the secondary centers must maintain a very good access to the main urban center, which requires low communication costs.

Trying to explain the emergence of cities with various sizes, our framework, unlike Helpman (1998), Tabuchi (1998) and others, allows cities to be polycentric. Moreover, in contrast to Sullivan (1986), Wieand (1987), and (Helsley and Sullivan, 1991), in our treatment, there are no pre-specified locations or numbers of subcenters, and our model is a fully closed general equilibrium spatial economy. As mentioned above, emergence of additional job centers is based on the urge towards decreasing of urban costs, rather than consumer's "propensity to big malls", as suggested by Anas and Kim (1996). Our approach, that takes into account various types of costs (trade, commuting, and communication) is similar to Cavailhès et al. (2007) with one important exception. We drop very convenient (yet non-realistic) assumption on "long narrow city." Our analysis is extended to the two-dimension because the geographical space in the real world is better approximated by a two-dimensional space.

2 Model overview

2.1 Spatial structure

Consider an economy with $G \geq 1$ regions, separated with physical distance, one sector and two primary goods, labor and land. Each region can be urbanized by accommodating firms and workers within a city, and is formally described by a two-dimensional space $X = \mathbb{R}^2$. Whenever a city exists, it has a central business district (in short CBD) located at the origin $0 \in X$ (see Figure 1a).

Firms are free to locate in the CBD or to set up in the suburbs of the metro where they form *secondary business districts*, SBD in short. Both the CBD and SBDs are assumed to be dimensionless. In what follows, the superscript C is used to describe variables related to the CBD, whereas S describes the variables associated with a SBDs. We consider the case where the CBD of urbanized region g is surrounded by $m_g \geq 0$ SBDs; $m_g = 0$ corresponds to the case of monocentric city. Without loss of generality, we focus on the only one of SBDs, because all SBDs are assumed to be identical.

Even though firms consume services supplied in each SBD, the higher-order functions (specific local public goods and non-tradable business-to-business services such as marketing, banking, insurance) are still located in the CBDs. Hence, for using such services, firms set up in a SBD must incur a communication cost, $K > 0$. In paper of Cavailhès et al. (2007) more general communication cost function $\mathcal{K}(x^S) = K + k \cdot ||x^S||$ was used, where $k > 0$, and $||x^S||$ is a distance between CBD and SBD. This generalization does not change the nature of our results, though analytical calculation became more tedious. Both the CBD and the SBD are surrounded by residential areas occupied by workers (see Figure 1a). There is no overlapping between residence zones. Furthermore, as the distance between the CBD and SBD is small compared to the intercity distance, we disregard the intra-urban transport cost of goods. Note that using the more general type of communication cost with $k > 0$ leads to consequence that in equilibrium Central and any Secondary residence zones should be *adjacent* to each other. This condition is non-necessary for fixed communication cost, although the real SBD can not be placed too far from City Center.

Under those various assumptions, the location, size and number of the SBDs as well as the size of the CBD will be endogenously determined. In other words, apart from the assumed existence of the CBD, the internal structure of each city is endogenous.

2.2 Workers/Consumers

The economy is endowed with L workers, distributed across the regions, where population of city g is l_g, i.e., $\sum_{g=1}^{G} l_g = L$. In this paper our primary focus is on the intra-city cost effects and on the trade, therefore the distribution of labor is considered as exogenous. The welfare of a worker depends on her consumption of the following three goods. The first good is unproduced and homogeneous. It is assumed to be costlessly tradable and chosen as the numéraire. The second good is produced as a continuum n of varieties of a horizontally differentiated good under monopolistic competition and increasing returns, using labor as the only input. Any variety of this good can be shipped from one city to the other at a unit cost of $\tau > 0$ units of the numéraire. The third good is land; without loss of generality, we set the opportunity cost of land to zero. Each worker living in city $1 \leq g \leq G$ consumes a residential plot of fixed size chosen as the unit of area. The worker also chooses a quantity $q(i)$ of variety $i \in [0, n]$, and a quantity q_0 of the numéraire. She is endowed with one unit of labor, which is supplied absolutely inelastically.

Preferences over the differentiated product and the numéraire are identical across workers and cities and represented by Ottaviano's quasi-linear utility function

$$U(q_0; q(i), i \in [0,n]) = \alpha \int_0^n q(i)\mathrm{d}i - \frac{\beta}{2}\int_0^n [q(i)]^2\mathrm{d}i - \frac{\gamma}{2}\left[\int_0^n q(i)\mathrm{d}i\right]^2 + q_0 \tag{1}$$

where $\alpha, \beta, \gamma > 0$. Demand for these products (provided that job and location are already chosen) is determined by maximizing of utility subject to the budget constraint

$$\int_0^n p(i)q(i)\mathrm{d}i + q_0 + R_g(x) + T_g(x) = w_g(x) + \frac{ALR_g}{l_g}, \tag{2}$$

where $R_g(x)$ is the land rent prevailing at location x, $T_g(x)$ is commuting cost, $w_g(x)$ is the wage, and $ARL_g = \int\limits_{x \in X} R_g(x)\mathrm{d}x$ is an *aggregated land rent* in the city g. This form of the budget constraint suggests that there are no landlords, who appropriate the land rent, moving it out of city budget. In other words, land is in a joint ownership of all citizen.

Each worker commutes to her employment center – without cross-commuting – and bears a unit commuting cost given by $t > 0$, so that for the worker located at x the commuting cost, $T_g(x)$, is either $t||x||$ or $t||x - x_g^S||$ according to the employment center. Moreover, the wage $w_g(x)$ depends only on type of employment center and takes one of two possible values: wage in CBD, w_g^C, or wage in SBD, w_g^S, which is uniform across all SBDs. Thus, the budget constraint of an individual working in the CBD is as follows

$$\int_0^n p(i)q(i)\mathrm{d}i + q_{0g} + R_g^C(x) + t||x|| = w_g^C + \frac{ALR_g}{l_g}, \tag{3}$$

while for individuals working in the SBD, located at x_g^S, it takes the form

$$\int_0^n p(i)q(i)\mathrm{d}i + q_{0g} + R_g^S(x) + t||x\text{-}x_g^S|| = w_g^S + \frac{ALR_g}{l_g}. \tag{4}$$

2.3 Firms

Our basic assumption on the manufacturing technology is that producing $q(i)$ units of variety i requires a given number φ of labor units. One may assume that producing one unit of variety i requires additionally $c \geq 0$ units of numéraire. This is not significant generalization, however, because this model is technically equivalent to one with $c = 0$ (see Ottaviano et al., 2002). Another option is that production of one unit of variety i requires $v > 0$ units of labor, thus the total labor requirement for the firm, producing $q(i)$ units of differentiated good, is $\varphi(q(i)) = f + v \cdot q(i)$, where f is a fixed production cost. This generalization will be considered in Section 5.

There is no scope economy so that, due to increasing returns to scale, there is a one-to-one relationship between firms and varieties. Thus, the total number of firms is given by $n = L/\varphi$. Labor market clearing implies that the number of firms located (or varieties produced) in city g is such that $n_g = \lambda_g n$, where $\lambda_g = l_g/L$ stands for the share of workers residing in g. Denote by Π_g^C (respectively Π_g^S) the profit of a firm set up in the CBD of city g (respectively the SBD). When the firm producing variety i is located in the CBD, its profit function is given by:

$$\Pi_g^C(i) = I_g(i) - \varphi \cdot w_g^C, \tag{5}$$

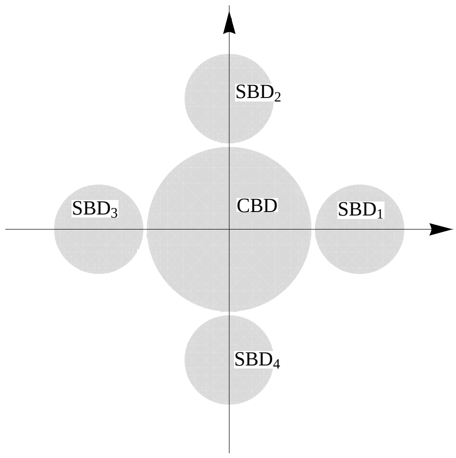
where

$$I_g(i) = p_{gg}(i) \cdot Q_{gg}(i) + \sum_{f \neq g} (p_{gf}(i) - \tau) \cdot Q_{gf}(i)$$

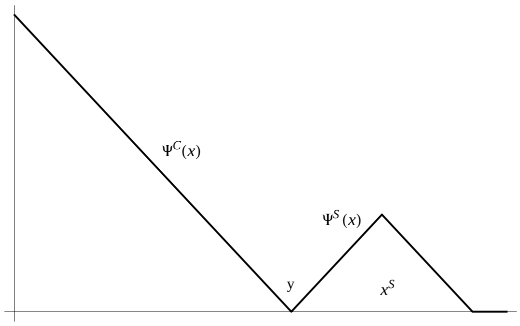
stands for the firm's revenue earned from local sales $Q_{gg}(i)$ and from exports $Q_{gf}(i)$ from city g to various cities f. When the firm sets up in the SBD of the same city, its profit function becomes:

$$\Pi_g^S(i) = I_g(i) - \varphi \cdot w_g^S - K. \tag{6}$$

The firm's revenue is the same as in the CBD because shipping varieties within the city is costless, so that prices and outputs do not depend on firm's location in the city.



a) Polycentric city with four SBDs



b) Bid rents in Center and in representative SBD

Figure 1: Polycentric City with CBD and four SBDs

3 Urban Costs and Decentralization within a City

A city equilibrium is such that each individual maximizes her utility subject to her budget constraint, each firm maximizes its profits, and markets clear. Individuals choose their workplace (CBD or SBD) and their residential location with respect to given wages and land rents. In each workplace, the equilibrium wages are determined by a bidding process in which firms compete for workers by offering them higher wages until no firm can profitably enter the market. Given such equilibrium wages and the location of workers, firms choose to locate either in the CBD or in the SBD. At the city equilibrium, no firm has an incentive to change place within the city. To ease the burden of notation, we drop the subscript g.

3.1 Land rents and Wage wedge

Let $\Psi^C(x)$ and $\Psi^S(x)$ be the bid rent at $x \in X$ of an individual working, respectively, in the CBD and in the representative SBD. Land is allocated to the highest bidder. An opportunity cost of land (e.g., for agricultural use) is assumed to be zero. Urban costs (commuting and communication) increase with Euclidean distance, thus "efficient" shapes of both Central and Secondary residence zones are circles tangent each other (see Figure 1a). All locations with the same distance to the *corresponding* Business District (Central or Secondary) are equivalent with respect to urban cots. Without loss of generality we assume that representative SBD, x^S, resides at abscissa, then the point of tangency, y, of residence zones, Central and representative Secondary, also belongs to abscissa. It implies that we can limit ourself to positive half of x-axis, which represents all possible values of the bid rent functions (see Figure 1b). Because there is only one type of labor, at the city equilibrium it must be that the housing rent $R(x) = \max\left\{\Psi^C(x), \Psi^S(x), 0\right\}$. Within each city, a worker chooses her location so as to maximize her utility $U(q_0, q(i); i \in [0, n])$ under the corresponding budget constraint, (3) or (4).

Because of the fixed lot size assumption, at the city equilibrium the value of the equilibrium consumption of the nonspatial goods

$$\int_0^n p(i)q(i)\mathrm{d}i + q_0 = E \tag{7}$$

is the same regardless of the worker's location:

$$w^C + \frac{ALR}{l} - R^C(x') - t||x'|| = E^C(x') \equiv E^S(x'') = w^S + \frac{ALR}{l} - R^S(x'') - t||x'' - x^S||$$

for all x', x'', belonging to CBD and SBD residence zones, respectively. To ensure this, we assume for now, that the share of firms located in the CBD, θ, is given, then $(1-\theta)/m$ is the share of firms in each SBD.

Proposition 1. *For any given city population l, SBD number m, and CBD share of firms θ:*

i) Central zone radius r^C and SBD zone radius r^S are as follows:

$$r^C = \sqrt{\frac{\theta l}{\pi}}, \quad r^S = \sqrt{\frac{(1-\theta)l}{m\pi}}. \tag{8}$$

ii) The following land rent function equalizes the disposable income E for all central and suburb residence locations x:

$$R(x) = t \cdot \max_{1 \le k \le m} \left\{0, \sqrt{\frac{\theta l}{\pi}} - ||x||, \sqrt{\frac{(1-\theta)l}{m\pi}} - ||x_k^S - x||\right\},$$

where $\left\{x_k^S\right\}_{k=1}^m$ is a set of all SBD locations.

iii) Redistributed aggregated land rent:

$$\frac{ALR}{l} = \frac{1}{l}\int_X R(x)dx = \frac{t}{3} \cdot \sqrt{\frac{l}{\pi}}\left[\theta^{3/2} + \frac{(1-\theta)^{3/2}}{\sqrt{m}}\right]. \tag{9}$$

iv) In equilibrium there exists the positive wage wedge between CBD and SBD

$$w^C - w^S = t \cdot \left(\sqrt{\frac{\theta l}{\pi}} - \sqrt{\frac{(1-\theta)l}{m\pi}} \right) \tag{10}$$

which is non-negative for all $\theta \in \left[\frac{1}{1+m}, 1 \right]$.

For analytical proof see Appendix. Figure 2 presents the plot of function $R(x)$ for $m = 4$:

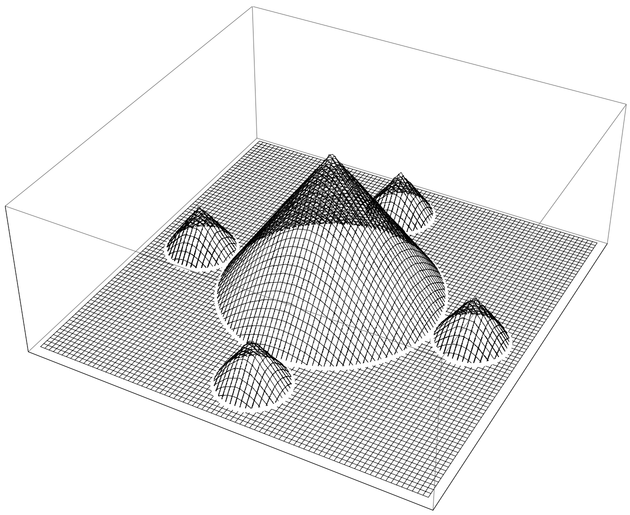


Figure 2: Rent function $R(x)$

In particular, rent function for CBD and representative SBD, x^S, are as follows

$$\begin{array}{rl} R^C(x) = \Psi^C(x) = t \cdot \left(\sqrt{\frac{\theta l}{\pi}} - ||x|| \right) & \text{for} \quad ||x|| \leq r^C = \sqrt{\frac{\theta l}{\pi}} \\ R^S(x) = \Psi^S(x) = t \cdot \left(\sqrt{\frac{(1-\theta)l}{m\pi}} - ||x^S - x|| \right) & \text{for} \quad ||x^S - x|| \leq r^S = \sqrt{\frac{(1-\theta)l}{m\pi}} \end{array} . \tag{11}$$

Their plots are central big cone and one of adjacent small cones, centered at specified location x^S.

3.2 Urban Costs

Let's define *urban cost* function as a sum of rent and commuting costs *minus* the individual share of aggregated land rent $\dfrac{ALR}{l}$.[1] Due to (9) and (11) these urban costs are as follows

$$
\begin{aligned}
C_u^C &= \Psi^C(x) + t||x|| - \frac{ALR}{l} = t\sqrt{\frac{\theta l}{\pi}} - \frac{t}{3}\cdot\sqrt{\frac{l}{\pi}}\left[\theta^{3/2} + \frac{(1-\theta)^{3/2}}{\sqrt{m}}\right], \\
C_u^S &= \Psi^S(x) + t||x - x^S|| - \frac{ALR}{l} = t\sqrt{\frac{(1-\theta)l}{\pi}} - \frac{t}{3}\cdot\sqrt{\frac{l}{\pi}}\left[\theta^{3/2} + \frac{(1-\theta)^{3/2}}{\sqrt{m}}\right].
\end{aligned}
\tag{12}
$$

The city equilibrium implies that the identity $w^C - C_u^C = w^S - C_u^S$ holds. In these terms, the *wage wedge* identity may be rewritten as a difference between urban costs in CBD and SBD: $w^C - w^S = C_u^C - C_u^S$.

3.3 Equilibrium city structure

Regarding the labor markets, the equilibrium wages of workers are determined by the zero-profit condition. In other words, operating profits are completely absorbed by the wage bill. Hence, the equilibrium wage rates in the CBD and in the SBDs must satisfy the conditions $\Pi^C(w^{C*}) = 0$ and $\Pi^S(w^{S*}) = 0$, respectively. Thus, setting (5) (respectively (6)) equal to zero, solving for w^{C*} (respectively w^{S*}), we get:

$$w^{C*} = \frac{I}{\varphi}, \quad w^{S*} = \frac{I-K}{\varphi} \tag{13}$$

Hence $w^{C*} - w^{S*} = \dfrac{K}{\varphi} > 0$, due to (8). Comparing the previous formula with (10) we obtain that CBD share of firms, θ satisfies the identity

$$\varphi t\sqrt{m\theta l} = K\sqrt{m\pi} + \varphi t\sqrt{(1-\theta)l}. \tag{14}$$

Admissible solution θ^* of equation (14) will be referred as *equilibrium CBD share*.

Proposition 2. *i) Let* $l \le \dfrac{\pi K^2}{\varphi^2 t^2}$ *then the unique solution of equation* (14) *is* $\theta^* = 1$ *with* $m = 0$*, i.e. city may be monocentric only;*

ii) Let $l > \dfrac{\pi K^2}{\varphi^2 t^2}$ *then for each* $m \ge 1$ *equation* (14) *has unique solution* $\theta^* \in \left(\dfrac{1}{1+m}, 1\right)$*, i.e. there exists a unique equilibrium SBD share of firms.*

[1] For technical reasons it is convenient to treat $\frac{ALR}{l}$ as some kind of rent compensation, subtracting it from costs rather adding to wage.

iii) The CBD share of firms θ^* decreases with respect to population l, number of SBDs m and commuting costs t. Moreover, θ^* increases with respect to communication cost K and

$$\lim_{l\to\infty}\theta^* = \lim_{t\to\infty}\theta^* = \lim_{K\to 0}\theta^* = \frac{1}{1+m}. \tag{15}$$

For analytical proof see Appendix.

Remark. Let $r^M(l) = \sqrt{\dfrac{l}{\pi}}$ and note that it is in fact a radius of *monocentric* city with population l. Inequality $l < \dfrac{\pi K^2}{\varphi^2 t^2}$ holds if and only if $\varphi t \cdot r^M(l) > K$. The left-hand side of this inequality is *total commuting costs* of firm's workers, residing at periphery of monocentric city in case of firm's location at CBD. To hire φ workers from periphery, firm should compensate their commuting costs in wage. On the other hand, locating the firm at the periphery causes the lesser communication cost K. Thus, producing on periphery (in SBD) is more efficient for new firm entering the industry. For any given K we obtain *minimum polycentric city population*: $l^P = \dfrac{\pi K^2}{\varphi^2 t^2}$. If city population $l \le l^P$ the corresponding central share $\theta^* \equiv 1$, i.e. city pattern is monocentric. It is not surprising that increasing in commuting costs t leads to lager dispersion of firms and workers. For very large magnitude of t, communication costs K become negligible and the distribution of production across all business centers is almost uniform.

Substituting equilibrium SBD share $\theta^*(m, l, t)$ into the urban cost function

$$C_u^C = t\sqrt{\frac{\theta l}{\pi}} - \frac{t}{3}\cdot\sqrt{\frac{l}{\pi}}\left[\theta^{3/2} + (1-\theta)\sqrt{\frac{1-\theta}{m}}\right]$$

and taking into account that

$$\sqrt{\frac{1-\theta^*}{m}} = \sqrt{\theta^*} - \frac{K\sqrt{\pi}}{\varphi\cdot t\sqrt{l}},$$

which follows from equation (14), we obtain that the urban cost function

$$C_u^C(l, m, t) = \frac{2t}{3}\sqrt{\frac{\theta^*(l, m, t)\cdot l}{\pi}} + \frac{K}{3\varphi}\cdot(1 - \theta^*(l, m, t)). \tag{16}$$

In particular,

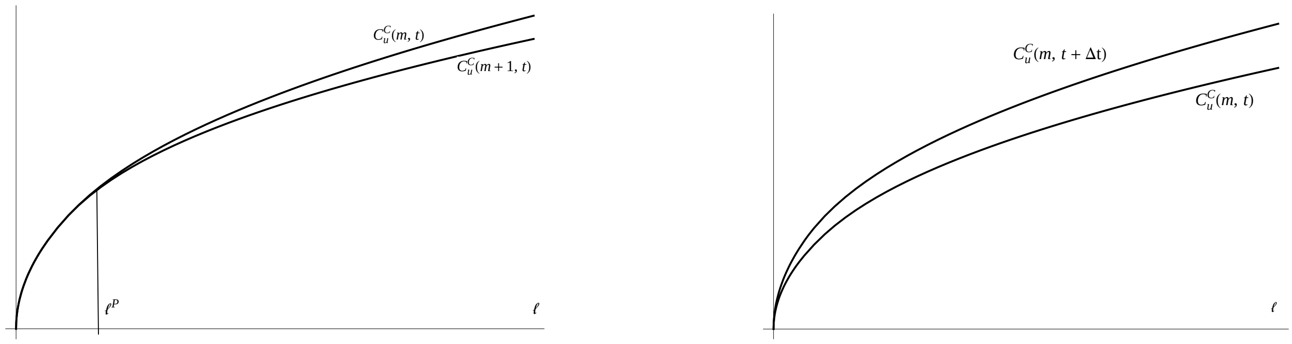


Figure 3: Comparative statics of urban costs

$$C_u^C(l, 0, t) = \frac{2}{3} t \sqrt{\frac{l}{\pi}} \text{ for } m = 0 \text{ and } l \geq 0$$
$$C_u^C(l, m, t) = \frac{2}{3} t \sqrt{\frac{l}{\pi}} \text{ for all } m > 0 \text{ and } l \leq \frac{\pi K^2}{\varphi^2 t^2},$$

because in these cases $\theta^* \equiv 1$.

Proposition 3. *Function $C_u^C(l, m, t)$ is continuous for all $m \geq 0$, $l \geq 0$, $t \geq 0$ and continuously differentiable function for $m > 0$, $l > 0$, $t > 0$. Moreover, $C_u^C(l, m, t)$ strictly increases with respect to l and t, strictly decreases with respect to m for all $l > l^P$.*

For analytical proof see Appendix. Figure 3 represents results of Proposition 3 in visual way as simulation in Wolfram's Mathematica 8.0.

Remark. Note that urban cost function C_u is *concave* with respect to l. It may reflect the fact that the housing price at periphery of residence zone increases with l sufficiently slow. The newcomers reside at the periphery, where the housing rent is very small. Moreover, unlike the linear model, in two-dimensional case this periphery *enlarges* as the city population grows. Though immigration increases competition for housing, an increment of the *per capita* urban costs C_u is less than before.

4 Inter-City Equilibrium

Until now we studied equilibrium decentralization within the city, or *Intra-City* equilibrium. Let's turn to *Inter-City* equilibrium assuming that the city populations l_g and numbers of SBD m_g are given for each city g. This paper focuses mainly on trade aspects, putting aside labor migration, therefore this assumption is consistent. Some considerations on endogenezation of SBD number are discussed at the end of this Section. Equilibrium shares of firms, θ_g^*, located at CBD, may be obtained independently, as solutions of equation (14) for each city g. These shares, in turn, allow to determine the urban costs C_{ug}, which do not depend on inter-city trade (and even on *existence* of other cities). On the contrary, wage

$$w_g^C = \frac{1}{\varphi}\left(p_{gg}(i)\cdot Q_{gg}(i) + \sum_{f\neq g}(p_{gf}(i)-\tau)\cdot Q_{gf}(i)\right),$$

substantially depends on trade, as well as consumer's utility $U(q_0; q(i), i\in[0,n])$. Moreover, if trade costs are too large, e.g., $\tau \geq p_{gf}(i)$, export is non-profitable and firms choose the domestic sales only, which implies

$$w_g^C = \frac{p_{gg}(i)\cdot Q_{gg}(i)}{\varphi}.$$

Now we split the study of equilibrium into two sub-cases: **Equilibrium under Autarchy** and **Equilibrium with Bilateral Trade**.

4.1 Equilibrium under Autarchy

This case suggests that equilibrium is separately established for each city, hence we may drop subscript g and consider the city with population l and SBD number m. Moreover, assume that the number of firms n is given and condition $w^C - C_u^C > 0$ holds. What determines n and how to provide this consumers' "surviving condition" will be discussed at the end of this subsection.

Representative consumer maximizes utility

$$U(q_0; q(i), i\in[0,n]) = \alpha\int_0^n q(i)\mathrm{d}i - \frac{\beta}{2}\int_0^n [q(i)]^2\mathrm{d}i - \frac{\gamma}{2}\left[\int_0^n q(i)\mathrm{d}i\right]^2 + q_0$$

subject to

$$\int_0^n p(i)q(i)\mathrm{d}i + q_0 = w^C - C_u^C,$$

First of all, recall some well-known results concerning consumer's problem with this form of utility.

Lemma 1. *Consumer's demand is linear function*

$$q(i) = \frac{\alpha}{\beta+\gamma n} - \frac{1}{\beta}p(i) + \frac{\gamma}{(\beta+\gamma n)\beta}\cdot P,$$

where $P = \int_0^n p(i)di$ is price index. Equilibrium prices and demand of representative are uniform by *goods*

$$p^*(i) \equiv p^* = \frac{\alpha\beta}{2\beta + \gamma n},\ q^*(i) \equiv q^* = \frac{\alpha}{2\beta + \gamma n}.$$

Consumer's surplus at equilibrium is equal to

$$CS = \frac{\alpha^2 n(\beta + \gamma n)}{2(2\beta + \gamma n)^2}.$$

For analytical proof see Appendix and/or Ottaviano et al. (2002). Using this lemma and taking into account that $n = \dfrac{l}{\varphi}$ we obtain the terms of equilibrium wage at CBD

$$w^{C*} = \frac{l \cdot p^* \cdot q^*}{\varphi} = \frac{\alpha^2 \beta \varphi l}{(2\beta\varphi + \gamma l)^2}$$

and consumer's surplus

$$CS = \frac{\alpha^2(\beta\varphi + \gamma l)l}{2(2\beta\varphi + \gamma l)^2},$$

which does not depend on consumer residence. Moreover sum of wage and consumer surplus (urban gains, for short) is

$$G_u^C = CS + w^{C*} = \frac{\alpha^2(3\beta\varphi + \gamma l)l}{2(2\beta\varphi + \gamma l)^2}.$$

Finally, consumer's welfare in CBD is a difference of urban gains and urban costs

$$V^C = CS + w^{C*} - C_u^C.$$

Similar to CBD we may calculate the corresponding SBD's characteristics: wage

$$w^{S*} = \frac{\alpha^2 \beta \varphi l}{(2\beta\varphi + \gamma l)^2} - t\left(\sqrt{\frac{\theta^* l}{\pi}} - \sqrt{\frac{(1-\theta^*)l}{m\pi}}\right),$$

urban gains

$$G_u^S = CS + w^{S*} = \frac{\alpha^2(3\beta\varphi + \gamma l)l}{2(2\beta\varphi + \gamma l)} - t\left(\sqrt{\frac{\theta^* l}{\pi}} - \sqrt{\frac{(1-\theta^*)l}{m\pi}}\right),$$

where θ^* is solution of equation (14). Note that indirect utility

$$V^S = CS + w^{S*} - C_u^S \equiv CS + w^{C*} - C_u^C = V^C.$$

Proposition 4. *Wage function* $w^{C*}(l)$ *strictly increases for all* $0 \le l < \dfrac{2\beta\varphi}{\gamma}$ *and strictly convex for* $l > \dfrac{2\beta\varphi}{\gamma}$. *Moreover,*

$$\lim_{l\to+\infty} w^{C*}(l) = 0,\ w^{C*}(0) = 0,\ \frac{\partial w^{C*}}{\partial l}(0) = \frac{\alpha^2}{2} < +\infty.$$

Urban gains $G_u^C(l)$ *strictly increase for all* $l \ge 0$,

$$\lim_{l\to+\infty} G_u^C(l) = \frac{\alpha^2}{2\gamma},\ G_u^C(0) = 0.$$

Proof of this proposition is straightforward from the formulas of $w^{C*}(l)$ and $G_u^C(l)$.

"Surviving" condition

It is obvious that city equilibrium is consistent only if disposable income $w^{C*}(l) - C_u^C(l,m,t) \ge 0$, which is called *Surviving condition*. Feasibility of this condition depends on magnitude of commuting cost t: wage function w^{C*} is bounded and does not depend on t, while urban cost $C_u^C(l,m,t)$ increases unrestrictedly with t. As result, very large commuting cost makes the city formation impossible.

Proposition 5. *Let inequality* $\dfrac{K}{\varphi} < \dfrac{3\alpha^2}{16\gamma}$ *holds, then for any commuting cost* $t \in \left(0, \dfrac{K}{\varphi}\sqrt{\dfrac{\pi\gamma}{2\beta\varphi}}\right)$ *and any given SBD number* $m \ge 0$ *there exist numbers* $0 < l_{\text{min}}(m,t) < l_{\text{max}}(m,t) < \infty$, *such that inequality* $w^C(l) - C_u^C(l,m,t) \ge 0$ *holds if and only if* $l_{\text{min}}(m,t) \le l \le l_{\text{max}}(m,t)$. *Moreover, if* $m' > m$, *then* $l_{\text{min}}(m',t) \equiv l_{\text{min}}(m,t) < l_{\text{max}}(m,t) \le l_{\text{max}}(m',t)$ *and* $l^P < l^* \Rightarrow l_{\text{max}}(m,t) \le l_{\text{max}}(m',t)$.

For analytical proof see Appendix.

Remark. Note that inequality $\dfrac{K}{\varphi} < \dfrac{3\alpha^2}{16\gamma}$ is equivalent to

$$\frac{\alpha^2}{8\gamma} = \max_{l\ge0} w^{C*}(l) > C_u^C(l^P,m,t) = \frac{2t}{3}\sqrt{\frac{l^P}{\pi}} \equiv \frac{2K}{3\varphi},$$

which implies that the maximum possible wage exceeds the urban costs in the city with minimum polycentric city population l^P. The lack of this condition means that the production transfer to SBD is ineffective, because *per employee* communication cost $\dfrac{K}{\varphi}$ is too large.

Increasing of m broadens interval $[l_{\text{min}}(m,t), l_{\text{max}}(m,t)]$ (to be more precise, l_{min} is not affected by changes in SBD number, while l_{max} increases with respect to m). Moreover, disposable income

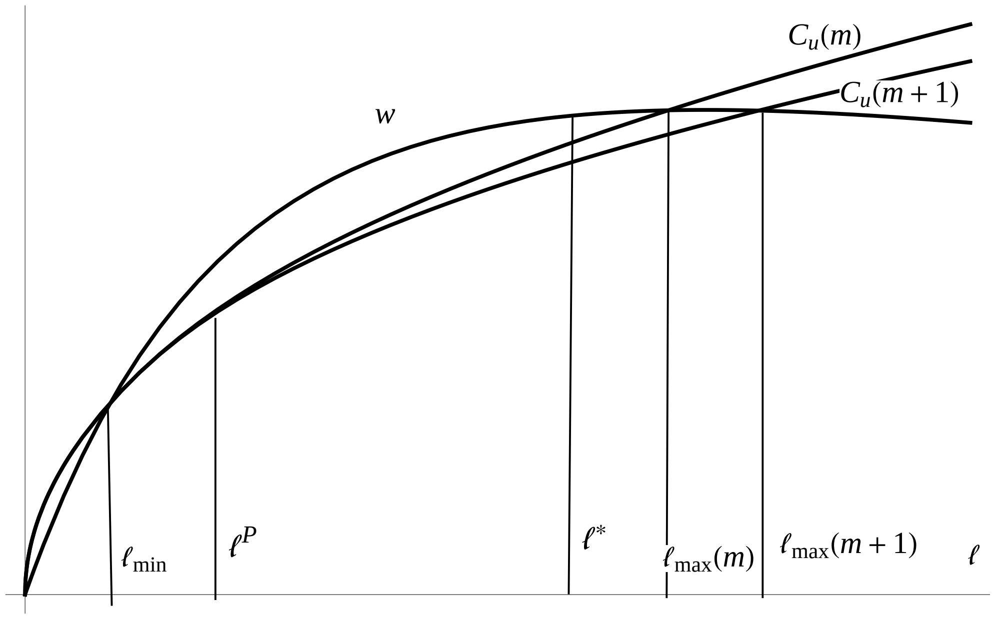


Figure 4: Comparative statics of the population limits

$w^C(l) - C_u^C(l, m, t)$ and welfare $V^C = G_u^C(l) - C_u^C(l, m, t)$ both increases with respect to m for all $l > l^P$. Figure 4 illustrates the equilibrium existence under autarchy and comparative statics of $l_{\max}$ with respect to m using simulation in Wolfram's Mathematica 8.0.

Remark. Previous considerations show that autarchy may be very restrictive to the city sizes: city survives only if its size exceeds the lower threshold $l_{\min} > 0$ and does not exceed the upper one $l_{\max}$. It is not surprising, because self-sufficient settlement of *industrial type* may exists only if its population is sufficiently large. Moreover, unrestrictedly growing urban costs (in particular, commuting cost) eventually stop the city growth. Developing of the city infrastructure (i.e. increasing in m) shifts up the *upper* bound $l_{\max}$, but cannot affect the *lower* critical point $l_{\min}$.

4.2 Endogenous SBD number

The concluding remark concerns the question: How to endogenize SBD number? There is no simple and unambiguous answer, because in practice it depends on many factors. One of the main questions is "Who can afford the building of additional suburb?" If answer is "None", we find ourself in setting with predefined number of SBDs (like model of Cavailhès et al., 2007). Otherwise, we assume that decision is up to 'City Developer', who takes into account the social welfare considerations. For example, when city population reaches the upper bound $l_{\max}$, an increasing the number of subcenters is urgently needed. Let's determine the following "compelled" SBD number for given population l and commuting cost t:

$$m^*(l,t) = \min\{m \mid l \le l_{\max}(m,t)\}.$$

Proposition 6. *SBD number m^* is non-decreasing function with respect to the city population l and commuting costs t, i.e., for all $l' > l$, $t' > t$ the following inequalities hold:*

$$m^*(l',t) \ge m^*(l,t),\ m^*(l,t') \ge m^*(l,t).$$

Proof. The statement concerning city population l is obvious: city is monocentric ($m^* = 0$) until population l exceeds $l_{\max}(0,t)$. By Proposition 5 upper bound $l_{\max}(1,t) > l_{\max}(0,t)$, thus while $l \le l_{\max}(1,m)$ the current SBD number $m^* = 1$, until l exceeds this upper bound, e.t.c. Increasing in commuting cost leads to decreasing of $l_{\max}(m,t) = \sup\left\{l \mid w^{C*}(l) \ge C_u^C(l,m,t)\right\}$, because $C_u^C(l,m,t)$ increases with respect to t by Proposition 3. Therefore, if $l > l_{\max}(m,t')$ for $t' > t$ then to recover surviving condition we need to increase SBD number until $l_{\max}(m',t') \ge l$.

Remark. Although this mechanism of endogenezation is not perfect, that theoretical comparative statics is fully supported by empirical evidences (see MacMillen and Smith, 2003). Anyway, it determines rather the endogenous minimum of SBD, which may be increased by some another reason, for example, to increase social welfare, i.e., total indirect utility of the city population.

Parametric Example

Consider the numerical example of how may change the inner structure of city under increasing of population size. Parameter values are chosen as follows: $\varphi = 5$, $K = 4$, $t = 1$, $\alpha = 6$, $\beta = 4$, $\gamma = 1$. Under these assumptions the lower bound of the city population $l_{\min} \approx 0.75$ is very small and once the city is grounded it starts to attract people, e.g., from rural neighborhood. Moreover, disposable income $w - C_u$ increases very quickly with respect to city size l at early stage, then it reaches the maximum and go down to zero when population size is close to upper population bound $l_{\max}(m)$. Its magnitude depends on city structure, i.e., number of SBD. For example, monicentric city reaches its maximum at $l_{\max}(0) \approx 97.5$, while for $m = 3$ the upper bound (or city capacity) is much larger, $l_{\max}(3) \approx 156.5$. The plots of disposable income for $m = 0, 1, 2, 3$ are presented at Figure 5a.

However, taking into account Consumer's Surplus along with Disposable Income we obtain that the resulting Consumer's Welfare (i.e., Indirect utility) $V = CS + w - C_u$ tends to grow further with

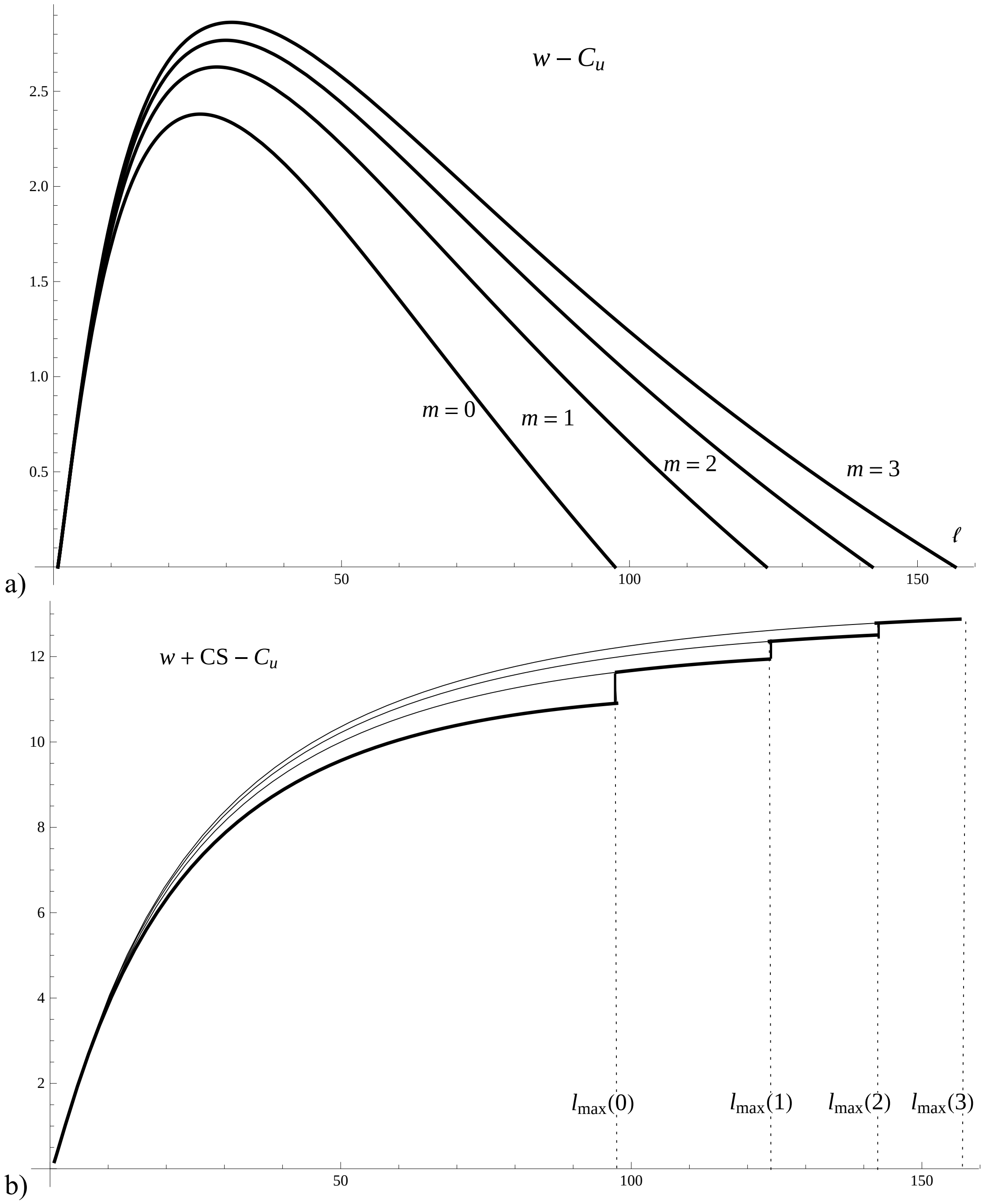


Figure 5: Disposable income and Welfare

respect to population size. It implies that there is a strong incentive for City Developer to increase the SBD number m, which in turn raises the city capacity. Of course, this expansion could be done "in advance", i.e., before the population size reaches the maximum. It is not so easy to predict, however, when it happens, thus the "cautious strategy" of City Developer is presented at Figure 5b by bold line, i.e., an additional SBD appears only if capacity of the city is exhausted. Moreover, we have assumed that the building of new SBD is costless, but this is not the case in real world. Thus, the expansion $m \to m+1$ will be well-grounded when *per capita* effect (welfare gap) reaches the maximum, i.e., at current $l_{\max}(m)$. It can be easily observed that this welfare leap quickly decreases with any next expansion of city structure, which eventually stops the increasing of the city population.

4.3 Bilateral Trade Equilibrium

The current subsection tell us what changes if trade comes to the place. To simplify description, assume that there are two cities, Home and Foreign. Let λ be the share of workers residing in Home city, then populations of both cities are $l_H = \lambda L$ and $l_F = (1-\lambda)L$, respectively. Moreover, the equilibrium masses of firms are $n_H = l_H/\varphi = \lambda \cdot n$, $n_F = l_F/\varphi = (1-\lambda) \cdot n$, where $n = L/\varphi$ is a total mass of firms in the world. Demands of Home representative consumer for domestic and imported differentiated goods, $q_{HH}(i)$ and $q_{FH}(i)$ respectively, are determined as solution of consumer problem

$$\max U(q_0; q(i), i \in [0, n_H + n_F])$$

subject to

$$\int_0^{n_H} p_{HH}(i) q_{HH}(i)\mathrm{d}i + \int_{n_H}^{n_H+n_F} p_{FH}(i) q_{FH}(i)\mathrm{d}i + q_0 = E_H = w_H^C - C_H^C. \tag{17}$$

Similarly demands of Foreign representative consumer, $q_{FF}(i)$ and $q_{HF}(i)$, are determined as solution of

$$\max U(q_0; q(i), i \in [0, n_H + n_F])$$

subject to

$$\int_0^{n_F} p_{FF}(i) q_{FF}(i)\mathrm{d}i + \int_{n_F}^{n_H+n_F} p_{HF}(i) q_{HF}(i)\mathrm{d}i + q_0 = E_F = w_F^C - C_F^C. \tag{18}$$

Facing these demands, firms maximize profits

$$I_H(i) = \lambda L \cdot p_{HH}(i) \cdot q_{HH}(i) + (1-\lambda)L \cdot [p_{HF}(i) - \tau] \cdot q_{HF}(i)$$
$$I_F(i) = (1-\lambda)L \cdot p_{FF}(i) \cdot q_{FF}(i) + \lambda L \cdot [p_{FH}(i) - \tau] \cdot q_{FH}(i)$$

and obtain optimal (equilibrium) prices and quantities. Zero-profit condition (13) determines equilibrium wages. It should be mentioned that bilateral trade is profitable only if trade costs τ are sufficiently small: $p_{HF}(i) > \tau$ and $p_{FH}(i) > \tau$. The following results are well-known, see, for example, original papers of Ottaviano et al. (2002) and Cavailhès et al. (2007).

Lemma 2. *Trade equilibrium prices are uniform by goods*

$$p^*_{HH}(i) \equiv p^*_{HH} = \frac{2\alpha\beta + \tau\gamma n_F}{2(2\beta+\gamma n)},\ p^*_{FF}(i) \equiv p^*_{FF} = \frac{2\alpha\beta + \tau\gamma n_H}{2(2\beta+\gamma n)},$$

$$p^*_{HF} = p^*_{FF} + \frac{\tau}{2},\ p^*_{FH} = p^*_{HH} + \frac{\tau}{2},$$

as well as equilibrium demands

$$q^*_{HH}(i) \equiv q^*_{HH} = \frac{1}{\beta+\gamma n}\left[\alpha - p^*_{HH} + \frac{\tau\gamma}{2\beta} n_F\right],\ q^*_{FH} = q^*_{HH} - \frac{\tau}{2\beta}$$

$$q^*_{FF}(i) \equiv q^*_{FF} = \frac{1}{\beta+\gamma n}\left[\alpha - p^*_{FF} + \frac{\tau\gamma}{2\beta} n_H\right],\ q^*_{HF} = q^*_{FF} - \frac{\tau}{2\beta}$$

Consumer's surplus

$$CS_H = \frac{\alpha^2 n}{2(\beta+\gamma n)} - \frac{\alpha}{\beta+\gamma n} \cdot [p^*_{HH} \cdot n_H + p^*_{FH} \cdot n_F] +$$

$$+\frac{1}{2\beta} \cdot \left[(p^*_{HH})^2 \cdot n_H + (p^*_{FH})^2 \cdot n_F\right] - \frac{\gamma}{2\beta \cdot (\beta+\gamma n)} \cdot [p^*_{HH} \cdot n_H + p^*_{FH} \cdot n_F]^2$$

Bilateral trade is profitable if $\tau < \tau_{trade} = \dfrac{2\alpha\beta}{2\beta+\gamma n}$.

For analytical proof see Appendix.

Substituting $\dfrac{\lambda L}{\varphi}$ for n_H, $\dfrac{(1-\lambda)L}{\varphi}$ for n_F and $\dfrac{L}{\varphi}$ for n we obtain the equilibrium prices and quantities for the Bilateral Trade Equilibrium. We focus on the Home city only, considerations for Foreign city are similar, *mutatis mutandis*. Without loss of generality, we may assume that $L \le l_{\max}(m_H)$, which implies, in particular, $w^{C*}_H(1) \ge C^C_u(1)$. It allow us to consider the whole unit interval $(0,1)$ as a set of admissible values for λ instead of truncation $(0, l_{\max}(m_H)/L)$.

Bilateral trade changes magnitudes of wage, consumer's surplus and indirect utility in comparison to autarchy case. To discriminate these cases, we add τ to notions of values, which are affected by trade. Recall that urban costs $C_u(\lambda)$ does not depend on τ. The following results are well-known (see, for example, Ottaviano et al. (2002) and Cavailhès et al. (2007)).

Lemma 3. *Home Equilibrium wage*

$$w_H^{C*}(\lambda,\tau) = \frac{\beta\varphi L}{(2\beta\varphi+\gamma L)^2}\left[\left(\alpha + \frac{\tau\gamma L}{2\beta\varphi}(1-\lambda)\right)^2 \cdot \lambda + \left((\alpha-\tau) - \frac{\tau\gamma L}{2\beta\varphi}(1-\lambda)\right)^2 \cdot (1-\lambda)\right] \tag{19}$$

is strictly concave function, increasing at $\lambda = 0$.

Home Consumer's Surplus

$$CS_H(\lambda,\tau) = \frac{\alpha^2 L}{2(\beta\varphi+\gamma L)} - \frac{\alpha L}{\beta\varphi+\gamma L}\cdot[p_{HH}^*\cdot\lambda + p_{FH}^*\cdot(1-\lambda)] +$$

$$+ \frac{L}{2\beta\varphi}\cdot\left[(p_{HH}^*)^2\cdot\lambda + (p_{FH}^*)^2\cdot(1-\lambda)\right] - \frac{\gamma L^2}{2\beta\varphi\cdot(\beta\varphi+\gamma L)}\cdot[p_{HH}^*\cdot\lambda + p_{FH}^*\cdot(1-\lambda)]^2 \tag{20}$$

is strictly increasing and concave function of λ.

Proof is straightforward (though tedious) from Lemma 2.

Proposition 7. *i) There exists $0 < \tau^* < \tau_{trade}$ such that for all $\tau \in (0,\tau^*)$ inequality $w^{C*}(\lambda) > C_u(\lambda)$ holds for all $\lambda \in (0,1)$.*

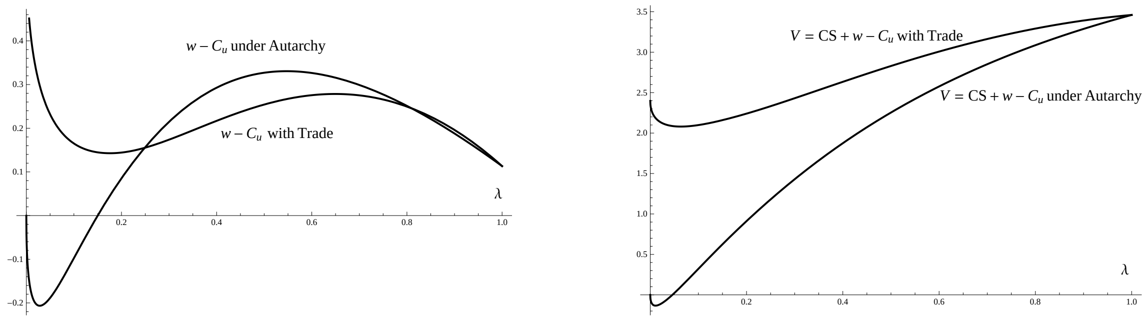
ii) There exists $0 < \tau^{**} < \tau_{trade}$ such that for all $\tau \in (0,\tau^{**})$ indirect utility with trade

$$V_H(\lambda,\tau) = CS_H(\lambda,\tau) + w_H^{C*}(\lambda,\tau) - C_{uH}^C(\lambda)$$

exceeds the corresponding utility under autarchy $V_H(\lambda) = CS_H(\lambda) + w_H^{C*}(\lambda) - C_{uH}^C(\lambda)$ for all $\lambda \in (0,1)$.

For analytical proof see Appendix. Typical results of simulation are presented at Figure 6.

Remark. Proposition 7(i) implies that sufficiently free trade cancels the lower bound of city size $l_{\min}$, i.e. small cities could survive, trading with the larger ones. It looks like small city became quasi-SBD for large one, replacing communication cost with trade cost. On the other hand, trade cannot cancel



a) Disposable Income with Trade and under Autarchy

b) Welfare with Trade and under Autarchy

Figure 6: Autarchy and Trade

the upper bound, or maximum city capacity. Thus, all considerations endogenous SBD number from subsection 4.2 are still valid. This proposition cannot be generalized for all $\tau \in (0, \tau_{\text{trade}})$. Computer simulations show that for τ sufficiently close to τ_{trade} both statements, (i) and (ii), are violated.

5 Equilibria under Variable Labor Input

Assume that production of one unit of variety i requires $v > 0$ units of labor additionally to the fixed input $f > 0$. Thus the total labor requirement for the firm, producing $Q(i)$ units of differentiated good, is $\varphi(Q(i)) = f + v \cdot Q(i)$. Similarly to previous consideration we consider two cases: Autarchy and Bilateral Trade.

5.1 Equilibrium under Autarchy

Note that Lemma 1 was proved regardless of the labor input type. Thus for city g with population l_g and number of firms n_g the equilibrium demand of representative consumer is $q_g^* = \frac{\alpha}{2\beta+\gamma n_g}$, while the total *autarchic* demand is $Q_g^* = l_g \cdot q_g^* = \dfrac{\alpha l_g}{2\beta + \gamma n_g}$. On the other hand, the labor market equilibrium condition implies that equilibrium mass of firms satisfies $n_g^* = \dfrac{l_g}{f + v \cdot Q_g^*}$. Thus n_g^*, as a (unique) positive solution of equation

$$n_g = \frac{l_g}{f + v \cdot \frac{\alpha l_g}{2\beta+\gamma n_g}} = \frac{2\beta l_g + \gamma l_g \cdot n_g}{(2\beta f + \alpha v l_g) + \gamma f \cdot n_g},$$

which is equivalent to

$$\gamma f \cdot n_g^2 + (2\beta f + \alpha v l_g - \gamma l_g) n_g - 2\beta l_g = 0.$$

Thus $n_g^* = n^*(l_g)$, where

$$n^*(l) = \frac{\beta}{\gamma}\left[(A \cdot l - 1) + \sqrt{(A \cdot l - 1)^2 + B \cdot l}\right], \tag{21}$$

where $A = \dfrac{\gamma - \alpha v}{2\beta f}$, $B = \dfrac{2\gamma}{\beta f}$. The only specific factor is the size of city population l_g. For the particular case $v = 0$ we obtain the well-known linear expression for the mass of firms $n_g^* = l_g/f$.

The following considerations are common for all cities, therefore, we drop the city subscript g.

Lemma 4. *Let $v > 0$, then $n^*(0) = 0$, $\dfrac{\partial n^*}{\partial l} > 0$, $\mathcal{E}_l(n^*) = \dfrac{l}{n^*}\dfrac{\partial n^*}{\partial l} < 1$, $\dfrac{\partial^2 n^*}{\partial l^2} < 0$. Moreover, $\lim\limits_{l\to\infty} n^*(l) = \infty$ for $v \le \dfrac{\gamma}{\alpha}$ and $\lim\limits_{l\to\infty} n^*(l) = \dfrac{2\beta}{\alpha v - \gamma}$ for $v > \dfrac{\gamma}{\alpha}$.*

Proof is straightforward from the formula (21).

Remark. The main difference from the case of fixed labor requirement is that for sufficiently large magnitude of variable labor input v the mass of firms is limited even if the city population size increases indefinitely. The following thought experiment may clarify this conclusion. Assume that new inhabitant comes to the city she increases demand for differentiated goods Q as well as supply of labor. Potentially, increasing in labor supply *positively* affects the mass of firms n. On the other hand, increasing in demand of all varieties of differentiated good require additional labor to produce this increment, to be more precise, it requires v workers per unit of variety. In turn, it *negatively* affects the mass of firms n. Thus, for sufficiently large values of v even the unrestricted labor inflow will be "consumed" by industrial sector without increasing in mass of firms. The following fact reflects the same tendencies for total demand of differentiated good.

Corollary. *Total demand $Q^*(l) = l \cdot q^*(n^*(l))$ is inelastic with respect to the population size l, i.e.,*

$$\mathcal{E}_l(Q^*) = \frac{l}{Q^*}\frac{\partial Q^*}{\partial l} = \frac{2\beta + \gamma \cdot (1 - \mathcal{E}_l(n^*(l)))\, n^*(l)}{2\beta + \gamma n^*(l)} \in (0, 1).$$

Now we can determine the equilibrium wages of CBD and SBD-employees in the city

$$w^{C*}(l) = \frac{l \cdot p^* \cdot q^*}{f + v \cdot l \cdot q^*} = \frac{\alpha^2 \beta n^*(l)}{(2\beta + \gamma n^*(l))^2},$$
$$w^{S*}(l) = \frac{\alpha^2 \beta n^*(l)}{(2\beta + \gamma n^*(l))^2} - t\left(\sqrt{\frac{\theta^* l}{\pi}} - \sqrt{\frac{(1-\theta^*) l}{m\pi}}\right),$$

as well as consumer's surpluses for both center and periphery

$$CS(l) = \frac{\alpha^2(\beta + \gamma n^*(l))n^*(l)}{2(2\beta + \gamma n^*(l))^2},$$

which does not depend on consumer residence. Moreover, sum of wage and surplus (urban gains, for short) is

$$G_u^C(l) = CS(l) + w^{C*}(l) = \frac{\alpha^2(3\beta + \gamma n^*(l)) \cdot n^*(l)}{2(2\beta + \gamma n^*(l))^2}.$$

Total Mass of Firms Under Autarchy

Let each of the cities $1 \leq g \leq G$ with population l_g established an equilibrium mass of firms $n_g = n^*(l_g)$. What is a total mass of firms $\sum_{g=1}^{G} n_g$ in comparison to $n^*(L)$ – the mass of firms in city with joint population $L = \sum_{g=1}^{G} l_g$? Note that in the case $v = 0$ we obtain

$$\sum_{g=1}^{G} n_g = \sum_{g=1}^{G} \frac{l_g}{f} = \frac{L}{f} = n^*(L),$$

i.e., total mass of firms does not depend on distribution of population. The case $v > 0$ is completely different.

Proposition 8. *Let $v > 0$, then $\sum_{g=1}^{G} n_g > n^*(L)$.*

For analytical proof see Appendix.

Remark. This result is quite natural, because Autarchy implies that local firms have no outer competitors and the total mass of firms under the more uniform distribution of population is greater then under total agglomeration. However, this variety of goods is almost useless for consumers, because only local varieties are accessible.

Urban costs

An equilibrium labor requirement per firm $\varphi_g^* = \varphi^*(l_g)$, where

$$\varphi^*(l) = f + \frac{\alpha v \cdot l}{2\beta + \gamma n^*(l)} = f + \frac{\alpha v \cdot l}{\beta \cdot \left[(Al + 1) + \sqrt{(Al - 1)^2 + B \cdot l}\right]}$$

An equilibrium CBD share of firms, θ_g^*, in city g satisfies the identity

$$\varphi^*(l_g) \cdot t\sqrt{m_g \theta l_g} = K\sqrt{m_g \pi} + \varphi^*(l_g) t\sqrt{(1-\theta) l_g},$$

Note that φ_g^* increases with respect to l_g. Indeed,

$$\varphi^*(l) = f + \frac{\alpha v \cdot l}{2\beta + \gamma n^*(l)} = f + \frac{\alpha v}{\frac{2\beta}{l} + \gamma \frac{n^*(l)}{l}}$$

and it is sufficient to prove that $\dfrac{n^*(l)}{l}$ decreases with respect to l:

$$\frac{\partial}{\partial l}\left(\frac{n^*(l)}{l}\right) = \frac{l \cdot \frac{\partial n^*}{\partial l} - n^*(l)}{l^2} = n^*(l) \cdot \frac{\mathcal{E}_l - 1}{l^2} < 0$$

due to Lemma 4. It implies that function $l \cdot (\varphi^*(l))^2$ strictly increases from 0 to ∞ and equation

$$l \cdot (\varphi^*(l))^2 = \frac{\pi K^2}{t^2} \iff l = \frac{\pi K^2}{(\varphi^*(l))^2 t^2}$$

has unique solution l^P.

Proposition 9. *i) Let $l_g \leq l^P$ then the unique equilibrium CBD share is $\theta_g^* = 1$ with $m_g = 0$, i.e. city may be monocentric only;*

ii) Let $l_g > l^P$ then for each $m \geq 1$ there exists the unique equilibrium CBD share $\theta_g^* \in \left(\frac{1}{1+m_g}, 1\right)$.

iii) The CBD share of firms θ_g^* decreases with respect to population l_g, number of SBDs m_g and commuting costs t. Moreover, θ_g^* increases with respect to communication cost K and

$$\lim_{l_g \to \infty} \theta_g^* = \lim_{t \to \infty} \theta_g^* = \lim_{K \to 0} \theta_g^* = \frac{1}{1+m_g}.$$

Proof of this statement is the same as one of Proposition 2 with the only difference in comparative statics with respect to l_g. See additional consideration to the proof of Proposition 2 in Appendix.

5.2 Trade Equilibrium for "Almost Equal" Cities

Once again turn to the case of the trade between two cities, Home and Foreign. Let λ be the share of workers residing in Home city, then populations of both cities are $l_H = \lambda L$ and $l_F = (1-\lambda)L$, respectively. Note that Lemma 2 was proved regardless of the labor input type. Thus for cities H and F with masses of firms n_H and n_F, correspondingly, the equilibrium prices are

$$p^*_{HH}(i) \equiv p^*_{HH} = \frac{2\alpha\beta + \tau\gamma n_F}{2(2\beta + \gamma n)},\ p^*_{FF}(i) \equiv p^*_{FF} = \frac{2\alpha\beta + \tau\gamma n_H}{2(2\beta + \gamma n)},$$

$$p^*_{HF} = p^*_{FF} + \frac{\tau}{2},\ p^*_{FH} = p^*_{HH} + \frac{\tau}{2},$$

where $n = n_H + n_F$, while equilibrium demands

$$q^*_{HH}(i) \equiv q^*_{HH} = \frac{1}{\beta + \gamma n}\left[\alpha - p^*_{HH} + \frac{\tau\gamma}{2\beta}n_F\right],\ q^*_{FH} = q^*_{HH} - \frac{\tau}{2\beta}$$

$$q^*_{FF}(i) \equiv q^*_{FF} = \frac{1}{\beta + \gamma n}\left[\alpha - p^*_{FF} + \frac{\tau\gamma}{2\beta}n_H\right],\ q^*_{HF} = q^*_{FF} - \frac{\tau}{2\beta}$$

Now the equilibrium masses of firms n^*_H, n^*_F may be obtained as solutions of the following equation system:

$$n_H = \frac{\lambda L}{f + v\cdot(\lambda L\cdot q^*_{HH} + (1-\lambda)L\cdot q^*_{HF})},\ n_F = \frac{(1-\lambda)L}{f + v\cdot(\lambda L\cdot q^*_{FH} + (1-\lambda)L\cdot q^*_{FF})}, \tag{22}$$

which are, in fact, the full employment conditions. Unfortunately, for $v > 0$ this system has no closed form solution, we can obtain such solution only for very specific values of λ.

Let $\lambda = 1$, then $n_F = 0$ and

$$n_H = \frac{L}{f + v\cdot L\cdot q^*_{HH}} = \frac{L}{f + \frac{v\cdot L}{\beta+\gamma n}\left(\alpha - \frac{2\alpha\beta+\tau\gamma n_F}{2(2\beta+\gamma n)} + \frac{\tau\gamma}{2\beta}n_F\right)} = \frac{L}{f + \frac{v\cdot L\cdot\alpha}{2\beta+\gamma n}}.$$

As result we obtain

$$n^*_H(1) = \frac{\beta}{\gamma}\left[(A\cdot L - 1) + \sqrt{(A\cdot L-1)^2 + B\cdot L}\right],$$

where $A = \dfrac{\gamma - \alpha v}{2\beta f}$, $B = \dfrac{2\gamma}{\beta f}$, which coincides with autarchy mass of firms. This is quite natural, because for $\lambda = 1$ there is no difference between Autarchy and Trade.

Now consider the case of equal cities, i.e., $\lambda = 1/2$. Equilibrium masses of firms $n_H^*(1/2)$, $n_F^*(1/2)$ are solutions of

$$n_H = \frac{L}{2f + v \cdot L \cdot (q_{HH}^* + q_{HF}^*)}, \; n_F = \frac{L}{2f + v \cdot L \cdot (q_{FH}^* + q_{FF}^*)}.$$

This system has the closed form solution

$$n_H^*(1/2) = n_F^*(1/2) = \frac{\beta}{2\gamma}\left[(A_\tau \cdot L - 1) + \sqrt{(A_\tau \cdot L - 1)^2 + B \cdot L}\right], \tag{23}$$

where $A_\tau = \dfrac{\gamma - \left(\alpha - \frac{\tau}{2}\right) v}{2\beta f} = A + \dfrac{\tau v}{4\beta f} > A$, $B = \dfrac{2\gamma}{\beta f}$.

Lemma 5. *Jacobian matrix J of the equation system (22) is non-degenerate at $\lambda_0 = 1/2$, $v_0 = 0$.*

Direct calculations show that Jacobian determinant of equation system

$$n_H \cdot (f + v \cdot (\lambda L \cdot q_{HH}^* + (1-\lambda) L \cdot q_{HF}^*)) = \lambda L, \; n_F \cdot (f + v \cdot (\lambda L \cdot q_{FH}^* + (1-\lambda) L \cdot q_{FF}^*)) = (1-\lambda) L,$$

which is equivalent to (22) is $\det J = \dfrac{f^2}{L^2} > 0$.

Lemma 5 and Implicit Function Theorem imply that there exists neighborhood Λ of (λ_0, v_0), such that system (22) is solvable for all $(\lambda, v) \in \Lambda$ and its solutions $n_H(\lambda)$, $n_F(\lambda)$ are differentiable functions of all parameters. Substituting these implicit functions into formulas for p_{ij}^* and q_{ij}^* we obtain the equilibrium prices and quantities.

Proposition 10. *There exists a neighborhood Λ of (λ_0, v_0), such that for all $(\lambda, v) \in \Lambda$ the masses of firms n_H and n_F satisfies the following conditions:*

i) $n_H^* + n_F^* > n_H^*(1)$; *ii)* $\dfrac{\partial n_H^*}{\partial \lambda} > 0$, $\dfrac{\partial n_F^*}{\partial \lambda} < 0$, $\dfrac{\partial n_H^*}{\partial L} > 0$, $\dfrac{\partial n_F^*}{\partial L} > 0$;
iii) $\dfrac{\partial n_H^*}{\partial f} < 0$, $\dfrac{\partial n_F^*}{\partial f} < 0$; *iv)* $\dfrac{\partial n_H^*}{\partial v} < 0$, $\dfrac{\partial n_F^*}{\partial v} < 0$; *v)* $\dfrac{\partial n_H^*}{\partial \tau} > 0$, $\dfrac{\partial n_F^*}{\partial \tau} > 0$.

For analytical proof see Appendix.

Remark. Note that most of this comparative statics is quite natural and predictable. Mass of firms is positive affected by local share of population λ and by total world population L, while both fixed f and variable v labor requirements affect negatively. Moreover, increasing in trade cost τ positively affects the mass of firms, because it weakens the competition with imported goods. The only novelty is that

$n_H^* + n_F^* > n_H^*(1)$ for $v > 0$ and λ sufficiently close to $1/2$. It means that the more uniform spatial distribution of population provides more varieties of differentiated good, then the total agglomeration. In other words, the more tough competition under total agglomeration decreases the total mass of firms in comparison to the more uniform spatial distribution of people. This effect appears only in model with non-zero variable labor requirement $v > 0$. If labor requirement does not depend on the produced quantity Q, the total mass of firms is independent on the labor distribution across cities. It should be mentioned also that the massive computer simulations allow to suggest that in fact $\Lambda = (0, 1) \times (0, +\infty)$. However, the general analytical tractability of this model is still questionable.

6 Conclusion

Paradigm of linear city is well suited for both actual "long narrow cities" and monocentric "two-dimensional", because in this case location may be characterized by scalar value – distance from Central Business District. In case of polycentricity – especially, with multiple Secondary Business Districts – linear model can't include all range of possibilities, being limited at most by two SBDs. Two-dimensional polycentric model, presented in this paper, lacks this disadvantage, while it is still tractable and intuitive. The results obtained in presented paper are of two kinds: some of them are common for both linear and two-dimensional models, while other are specific for two-dimensional model with several Secondary Business Districts. We discuss here these results, focusing on the specific ones.

Proposition 2 on Existence and Uniqueness of equilibrium CBD share implies that polycentric structure may exists only if population of city exceeds the certain threshold, i.e., too small city cannot bear the burden of polycentricity. This natural result is not 2D specific, nevertheless, it contains the statement that city with population beyond this threshold, could have any number of SBDs. Moreover, increasing in this number implies that *per capita* urban costs strictly decrease (see Proposition 3). It results in increasing (*ceteris paribus*) of disposable income and indirect utility of the city residents, therefore, developing of the inner city structure may be an important policy instrument.

It is obvious, that positiveness of disposable income is necessary condition for city residents. One of results obtained in this paper is that disposable income is positive if and only if city population is not less than strictly positive lower threshold a do not exceeds the finite upper bound (see Proposition 5). It means that the effective production (with increasing return to scale) cannot be developed on the base of too small settlement, and, *vice versa*, very large city cannot survive because of too heavy burden of urban costs. Increasing in SBD number shifts up the upper threshold (i.e., increases city capacity), therefore, extensive development of the city structure can be an effective policy instrument for sufficiently large cities (see Proposition 5). It cannot help, however, small cities to survive as industrial settlements.

Changes in city structure is mainly an instrument of inner policy, while change in trade openness may results outwards. Moreover, sufficiently high level of trade openness (i.e., sufficiently small trade costs) *shifts down to zero* the lower threshold of city population (see Proposition 6). It means that under condition of almost free trade, small cities could survive as satellites of large ones. Another benefit of *sufficiently free* trade is that *real* wage (indirect utility) increases for residents in all cities, not depending on their sizes (see Proposition 6), although this effect is more significant for small cities. It increases the relative attractiveness for the labor inflow. This inflow may result in overpopulation of city with given number of SBDs. To avoid this overpopulation, City Developer may increase the current SBD number, which increases city capacity. Mechanism of determining of endogenous *minimum* SBD number was suggested in Section 4.2, which is consistent with empirical evidences (see Proposition 7).

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7 APPENDIX

Table of Symbols:

- t – marginal commuting costs
- K – communication costs
- $f > 0$ – fixed labor requirements
- $v \geq 0$ – marginal labor requirements
- φ – total labor requirements per firm ($\varphi = f + v \cdot Q$ for $v > 0$)
- τ - trade costs
- L – overall population
- G – number of regions, $g \in \{1, \ldots G\}$

Subscript g may be dropped for separated city

- l_g – population of city g
- m_g – number of SBDs in city g
- n_g – mass of firms in city g

$d \in \{C, S\}$ – type of business district (Central or Secondary)

- $x_g^S \in X$ – location of (representative) SBD in city g
- w_g^d – wage of workers in district d of city g
- $\Pi_r^d(i)$ – profit of firms in district d of city r
- $R_g^d(x)$ – rent function in district d of city g
- $C_{ug}^d(l_g, m_g)$ – urban costs in district d of city g
- θ_g – share of firms in CBD of city g

Proof of Proposition 1

The land supply in equilibrium should equalize (inelastic) land demand

$$\pi \cdot \left(r^C\right)^2 + m \cdot \pi \cdot \left(r^S\right)^2 = l \cdot 1,$$

where $r^C$is radius of central zone, r^S is radius of single suburb. On the other hand, for given CBD's share of firms, θ, the labor market clearing in CBD (without cross-commuting) implies $\pi \cdot \left(r^C\right)^2 = \theta l$. Therefore,

$$y = r^C = \sqrt{\frac{\theta l}{\pi}}, \quad r^S = \sqrt{\frac{(1-\theta)l}{m\pi}}, \quad ||x^S|| = r^C + r^S = \sqrt{\frac{\theta l}{\pi}} + \sqrt{\frac{(1-\theta)l}{m\pi}}.$$

The budget constraint of an individual residing at point x and working in the CBD implies that

$$E^C(x) = w^C + \frac{ALR}{l} - \Psi^C(x) - t||x||,$$

whereas the budget constraint of an individual working in the SBD is

$$E^S(x) = w^S + \frac{ALR}{l} - \Psi^S(x) - t||x - x^S||.$$

Note that equalizing condition $E^C(x) \equiv E^S(x) \equiv \text{const}$ implies $\Psi^C(x) = A_1 - t||x||$, $\Psi^S(x) = A_2 - t||x - x^S||$, where A_1, A_2 do not depend on x. On the other hand, worker living at the border of the CBD residential area (i.e., at the point $y = r^C$ of the SBD residential area closest to CBD, see Figure 1b) is indifferent to the decisions of working in the CBD or in the SBD. Moreover, for the border location y an identities $\Psi^C(y) = \Psi^S(y) = 0$ hold, because there is no difference for landlord where to rent out this plot of land: to Central city, to Suburb or for agricultural use. Therefore,

$$A_1 = ty = t\sqrt{\frac{\theta l}{\pi}},\ A_2 = t \cdot (x^S - y) = t\sqrt{\frac{(1-\theta)l}{m\pi}}.$$

As result, we obtain

$$\frac{ALR}{l} = \frac{1}{l}\int\limits_X R(x)\mathrm{d}x = \frac{t}{3} \cdot \sqrt{\frac{l}{\pi}}\left[\theta^{3/2} + \frac{(1-\theta)^{3/2}}{\sqrt{m}}\right].$$

Note that no need to integrate actually this function. We may simply apply the well-known formula of the cone volume $V = \frac{1}{3}\pi h \cdot r^2$, where h is a hight and r is a radius of the base of cone.

Moreover, an identity $E^C(y) - E^S(y) = 0$ implies $w^C - w^S = A_1 - A_2 = t \cdot \left(\sqrt{\frac{\theta l}{\pi}} - \sqrt{\frac{(1-\theta)l}{m\pi}} \right)$. It means that the difference in the wages paid in the CBD and in the SBD compensates exactly the worker for the difference in the corresponding commuting costs. The wage wedge $w^C - w^S$ is positive as long as $\theta > \frac{1}{1+m}$, thus implying that the size of the CBD exceeds the size of each SBD.

Proof of Proposition 2

There is one-to-one correspondence between $\theta \in [0,1]$ and $\alpha \in [0, \frac{\pi}{2}]$ given by $\theta = \cos^2\alpha$. Substituting it into equation

$$\varphi t \sqrt{\frac{\theta l}{\pi}} = K + \varphi t \sqrt{\frac{(1-\theta)l}{m\pi}}$$

we obtain, after simple transformations, the following one

$$F(\alpha, l, m, t) := \cos\alpha - \frac{\sin\alpha}{\sqrt{m}} - \frac{K\sqrt{\pi}}{\varphi t \sqrt{l}} = 0. \tag{24}$$

Note that,

$$\frac{K\sqrt{\pi}}{\varphi t \sqrt{l}} < 1 \iff l > l^P = \frac{\pi K^2}{\varphi^2 t^2}$$

and

$$\frac{\partial F}{\partial \alpha} = -\sin\alpha - \frac{\cos\alpha}{\sqrt{m}} < 0.$$

Consider three possible cases:

i) $l < l^P$ then $F(\alpha, l, m, t) < (\cos\alpha - 1) - \frac{\sin\alpha}{\sqrt{m}} < 0$ and equation (24) has no roots.

ii) $l = l^P$ then $F(\alpha, l, m, t) = (\cos\alpha - 1) - \frac{\sin\alpha}{\sqrt{m}} = 0$ if and only if $\cos\alpha = 1$, which implies $\theta^* = 1$.

iii) $l > l^P$ then $F(0, l, m, t) = 1 - 0 - \frac{K\sqrt{\pi}}{\varphi t\sqrt{l}} > 0$ and $F(\frac{\pi}{2}, m, l) = 0 - \frac{1}{\sqrt{m}} - \frac{K\sqrt{\pi}}{\varphi t \sqrt{l}} < 0$. Thus there exists unique root $\alpha^* \in \left[0, \frac{\pi}{2}\right]$ of equation(24) and $\theta^* = \cos^2\alpha^*$.

Accordingly to Theorem on Implicit Function Derivative, we obtain

$$\frac{\partial \alpha^*}{\partial l} = -\frac{\partial F/\partial l}{\partial F/\partial \alpha} = \frac{K\sqrt{\pi} \cdot l^{-\frac{3}{2}}}{2\varphi t(\sin\alpha + \frac{\cos\alpha}{\sqrt{m}})} > 0.$$

It implies that $\theta^*(l) = \cos^2(\alpha^*(l))$ is *decreasing* function. Similarly,

$$\frac{\partial \alpha^*}{\partial m} = -\frac{\frac{\partial F}{\partial m}}{\frac{\partial F}{\partial \alpha}} = \frac{\sin\alpha \cdot m^{-\frac{3}{2}}}{2(\sin\alpha + \frac{\cos\alpha}{\sqrt{m}})} > 0,$$

thus $\theta^*(m) = \cos^2(\alpha^*(m))$ is also *decreasing* function. Furthermore,

$$\frac{\partial \alpha^*}{\partial t} = -\frac{\partial F/\partial t}{\partial F/\partial \alpha} = \frac{K\sqrt{\pi}\cdot l^{-\frac{1}{2}}}{\varphi t^2\left(\sin\alpha + \frac{\cos\alpha}{\sqrt{m}}\right)} > 0,$$

thus $\theta^*(t) = \cos^2(\alpha^*(t))$ is also *decreasing* function with respect to t. Finally,

$$\frac{\partial \alpha^*}{\partial K} = -\frac{\partial F/\partial t}{\partial F/\partial \alpha} = -\frac{\sqrt{\pi}}{\varphi t\sqrt{l}\cdot\left(\sin\alpha + \frac{\cos\alpha}{\sqrt{m}}\right)} < 0,$$

thus $\theta^*(t) = \cos^2(\alpha^*(t))$ *increases* with respect to t.

To obtain the limit value of θ^* is sufficient to note that equation (24) for $l \to \infty$, $t \to \infty$, $K \to 0$ transforms into

$$\cos\alpha - \frac{\sin\alpha}{\sqrt{m}} = 0$$

which is equivalent to

$$m \cdot \cos^2\alpha = \sin^2\alpha = 1 - \cos^2\alpha,$$

implying $\theta^* = \cos^2\alpha^* = \dfrac{1}{1+m}$. On the other hand, $K \to \infty$ implies $l^P \to \infty$, therefore $m = 0$ and $\theta^* = 1$ is a unique outcome.

Additional considerations for Proposition 9

The fact that φ is an increasing function of l does not affect almost nothing, except the comparative statics with respect to l. Indeed,

$$F(\alpha, l, m, t) := \cos\alpha - \frac{\sin\alpha}{\sqrt{m}} - \frac{K\sqrt{\pi}}{t\varphi(l)\sqrt{l}}$$

increases with respect to l, because $\varphi(l)$ and $\sqrt{l}$ are positive increasing functions of l. Therefore,

$$\frac{\partial \alpha^*}{\partial l} = -\frac{\partial F/\partial l}{\partial F/\partial \alpha} > 0$$

and $\theta^* = \cos^2\alpha^*$ decreases with respect to l.

Proof of Proposition 3

Let $y(l) = \theta^*(l) \cdot l$, then y is an implicit function defined by equation

$$G(y,l) = \sqrt{y} - \frac{1}{\sqrt{m}}\sqrt{l-y} - \frac{K\sqrt{\pi}}{\varphi t} = 0$$

which is equivalent to equation (15). Thus

$$\frac{\partial(\theta^*(l)\cdot l)}{\partial l} = -\left(\frac{\partial G}{\partial l}\Big/\frac{\partial G}{\partial y}\right) = \frac{\sqrt{y}}{\sqrt{m(l-y)}+\sqrt{y}} > 0,$$

moreover $\dfrac{\partial\theta^*}{\partial l} < 0$ by Proposition 2. It implies that function

$$C_u^C(l,m,t) = \frac{2t}{3}\sqrt{\frac{\theta^*(l,m,t)\cdot l}{\pi}} + \frac{K}{3\varphi}\cdot(1-\theta^*(l,m,t)).$$

increases with respect to l. Let's prove that $C_u^C(l)$ is continuously differentiable at $l = l^P = \dfrac{\pi K^2}{\varphi^2 t^2}$. Indeed, for all $l < l^P$ the urban cost function $C_u^C(l) = \dfrac{2t}{3}\sqrt{\dfrac{l}{\pi}}$, hence

$$\frac{\partial C_u^C}{\partial l}(l^P - 0) = \frac{\varphi t^2}{3\pi K}.$$

Note that $\theta^*(l^P) = 1$ and

$$\frac{\partial(\theta^*(l)\cdot l)}{\partial l}(l^P+0) = \frac{\sqrt{l^P}}{\sqrt{m(l^P-l^P)}+\sqrt{l^P}} = 1,$$

on the other hand,

$$\frac{\partial(\theta^*(l)\cdot l)}{\partial l}(l^P+0) = l^P\frac{\partial\theta^*}{\partial l}(l^P+0) + \theta^*(l^P).$$

It implies that $\dfrac{\partial\theta^*}{\partial l}(l^P+0) = 0$, therefore

$$\frac{\partial C_u^C}{\partial l}(l^P+0) = \frac{2t}{3}\cdot\frac{\partial\left(\sqrt{\frac{\theta l}{\pi}}\right)}{\partial l}(l^P+0) = \frac{\varphi t^2}{3\pi K} = \frac{\partial C_u^C}{\partial l}(l^P-0).$$

Recall that for $l \leq l^P$ the urban costs $C_u^C = \frac{2t}{3}\sqrt{\frac{\theta l}{\pi}}$, therefore $\frac{\partial C_u^C}{\partial m} \equiv 0$. Moreover, for $l > l^P$

$$\frac{\partial C_u^C}{\partial m} = \frac{\partial C_u^C}{\partial \theta}\frac{\partial \theta^*}{\partial m},$$

where $\frac{\partial \theta^*}{\partial m} < 0$ by Proposition 2 and

$$\frac{\partial C_u^C}{\partial \theta} = \frac{t}{3\sqrt{\theta}}\sqrt{\frac{l}{\pi}}\left(1 - \frac{\sqrt{\pi}K}{\varphi t\sqrt{l}}\sqrt{\theta}\right) > 0$$

because $l > l^P = \frac{\pi K^2}{\varphi^2 t^2}$ and $\theta < 1$. Therefore

$$\frac{\partial C_u^C}{\partial m} = \frac{\partial C_u^C}{\partial \theta} \cdot \frac{\partial \theta^*}{\partial m} < 0.$$

Moreover, $\theta^*(l^P) = 1$, hence

$$\frac{\partial C_u^C}{\partial \theta}(l^P + 0) = \frac{t}{3}\sqrt{\frac{l^P}{\pi}}\left(1 - \frac{\sqrt{\pi}K}{\varphi t\sqrt{l^P}}\right) = 0,$$

which implies that

$$\frac{\partial C_u^C}{\partial m}(l^P + 0) = \frac{\partial \theta^*}{\partial m} \cdot \frac{\partial C_u^C}{\partial \theta}(l^P + 0) = 0,$$

i.e. the urban cost function is *continuously* differentiable with respect to m.

Let $y(t) = \theta^*(t) \cdot t^2$, then y is an implicit function defined by equation

$$H(y, t) = \sqrt{my} - \sqrt{t^2 - y} - \frac{K\sqrt{m\pi}}{\varphi\sqrt{l}} = 0$$

which is equivalent to equation (15). Moreover,

$$\frac{\partial H}{\partial y} = \frac{\sqrt{m}}{2\sqrt{y}} + \frac{1}{2\sqrt{t^2 - y}} > 0, \quad \frac{\partial H}{\partial t} = -\frac{t}{2\sqrt{t^2 - y}} < 0,$$

therefore

$$\frac{\partial y}{\partial t} = -\left(\frac{\partial H}{\partial t}\bigg/\frac{\partial H}{\partial y}\right) > 0.$$

It implies that function $t \cdot \sqrt{\theta^*(t)} = \sqrt{y(t)}$ increases with respect to t, as well as $1 - \theta^*(l, m, t)$.

Therefore, urban costs function

$$C_u^C(l,m,t) = \frac{2t}{3}\sqrt{\frac{\theta^*(l,m,t)\cdot l}{\pi}} + \frac{K}{3\varphi}\cdot(1-\theta^*(l,m,t))$$

also increases with respect to t, increase with respect to t.

Proof of Lemma 1

From budget constraint (7)

$$q_0 = E - \int\limits_0^n p(i)q(i)\mathrm{d}i,$$

substituting it into utility (1) we obtain the following FOC:

$$\frac{\partial U}{\partial q(i)} = \alpha - \beta q(i) - \gamma \int\limits_0^n q(i)\mathrm{d}i - p(i) = 0, \tag{25}$$

and after integrating:

$$\alpha n - (\beta + \gamma n)\int\limits_0^n q(i)\mathrm{d}i - \int\limits_0^n p(i)\mathrm{d}i = 0. \tag{26}$$

Let

$$P = \int\limits_0^n p(i)\mathrm{d}i$$

denote price index, then (27) and (28) imply

$$q(i) = \frac{\alpha}{\beta+\gamma n} - \frac{1}{\beta}p(i) + \frac{\gamma}{(\beta+\gamma n)\beta}\cdot P.$$

Now turn to firm's problem $p(i)q(i) \to \max$, from the corresponding FOC we obtain that equilibrium price

$$p^*(i) \equiv p^* = \frac{\beta}{2}\left[\frac{\alpha}{\beta+\gamma n} + \frac{\gamma}{(\beta+\gamma n)\beta}\cdot P\right].$$

On the other hand,

$$P = \int\limits_0^n p^*(i)\mathrm{d}i = np^*,$$

thus equilibrium prices and quantities are, respectively,

$$p^*(i) \equiv p^* = \frac{\alpha\beta}{2\beta + \gamma n}, \; q^*(i) \equiv q^* = \frac{\alpha}{2\beta + \gamma n}.$$

Consumer's surplus for this equilibrium with linear demand may calculated standardly and it is equal to

$$CS = \frac{1}{2}\left(\frac{\alpha^2 n}{(\beta + \gamma n)} + \frac{n}{(\beta + \gamma n)}(p^*)^2\right) - \frac{\alpha n}{\beta + \gamma n} p^*.$$

Substituting the previous term for p^* we obtain that

$$CS^* = \frac{\alpha^2 n(\beta + \gamma n)}{2(2\beta + \gamma n)^2}.$$

For more details see original papers of Ottaviano et al. (2002) or Cavailhès et al. (2007).

Proof of Proposition 5

Note that wage $w^{C*}(l)$ is bounded function, while urban costs increase unrestrictedly with respect to l, hence, $w^{C*}(l) - C_u^C(l, m) < 0$ for all *sufficiently large* l. Moreover, $w^C(0) = C_u^C(0, m) = 0$, while $\frac{\partial w^{C*}}{\partial l}(0) = \frac{\alpha^2}{2} < \frac{\partial C_u^C}{\partial l}(0, m) = +\infty$, thus $w^{C*}(l) - C_u^C(l, m) < 0$ for all *sufficiently small* $l > 0$. It implies that set of l guaranteeing "the surviving condition" $w^{C*}(l) - C_u^C(l, m, t) \geq 0$ is subset of some interval $[l_{\text{min}}(m, t), l_{\text{max}}(m, t)]$, where $l_{\text{min}}(m, t) = \inf\left\{l > 0 \mid w^{C*}(l) - C_u^C(l, m, t) \geq 0\right\} > 0$ and $l_{\text{max}}(m, t) = \sup\left\{l > 0 \mid w^{C*}(l) - C_u^C(l, m, t) \geq 0\right\} < \infty$. It remains to prove that this subset is nonempty and inequality $w^{C*}(l) - C_u^C(l, m, t) \geq 0$ holds for all $l \in [l_{\text{min}}(m, t), l_{\text{max}}(m, t)]$, at least for $t \in \left(0, \frac{K}{\varphi}\sqrt{\frac{\pi\gamma}{2\beta\varphi}}\right)$.

Note that

$$t < \frac{K}{\varphi}\sqrt{\frac{\pi\gamma}{2\beta\varphi}} \Rightarrow t < \frac{3\alpha^2}{16\gamma}\sqrt{\frac{\pi\gamma}{2\beta\varphi}} \iff w^{C*}(l^*) = \frac{\alpha^2}{8\gamma} > \frac{2t}{3}\sqrt{\frac{\pi\gamma}{2\beta\varphi}} = C_u^C(l^*, 0, t) \geq C_u^C(l^*, m, t)$$

for all $m \geq 0$, where $l^* = \frac{2\beta\varphi}{\gamma}$ is the "maximum wage" population size. It implies that "surviving" set of city population is non-empty and $l_{\text{max}}(m, t) > l^*$. Moreover, inequality $\frac{K}{\varphi} < \frac{3\alpha^2}{16\gamma}$ ensures that equation

$$w^{C*}(l) = \frac{\alpha^2\beta\varphi l}{(2\beta\varphi + \gamma l)^2} = \frac{2K}{3\varphi}$$

has two real positive roots

$$l_{1,2} = \frac{\left(\frac{3\alpha^2\varphi^2\beta}{2K} - 4\beta\gamma\varphi\right) \mp \sqrt{\left(\frac{3\alpha^2\varphi^2\beta}{2K} - 4\beta\gamma\varphi\right)^2 - 16\beta^2\gamma^2\varphi^2}}{2\gamma^2}$$

and $w^{C*}(l) > \dfrac{2K}{3\varphi}$ if and only if $l_1 < l < l_2$. In particular, $l^* = \dfrac{2\beta\varphi}{\gamma} \in (l_1, l_2)$ because $w^{C*}(l^*) = \max w^{C*}$.

Note that

$$t < \frac{K}{\varphi}\sqrt{\frac{\pi\gamma}{2\beta\varphi}} = \frac{K}{\varphi}\sqrt{\frac{\pi}{l^*}} \Rightarrow t < \frac{K}{\varphi}\sqrt{\frac{\pi}{l_1}} \iff l^P = \frac{\pi K^2}{\varphi^2 t^2} > l_1.$$

Let $l_1 < l^P < l_2$ then inequality $w^{C*}(l^P) > \dfrac{2K}{3\varphi} = C_u^C(l^P, m)$ holds, which implies that $l_{\text{min}}(m,t) < l^P$. On the other hand, if $l^P \geq l_2 > l^*$ then $w^{C*}(l^*) > \dfrac{2K}{3\varphi} = C_u^C(l^P, m) > C_u^C(l^*, m)$, which also implies that $l_{\text{min}}(m,t) < l^P$.

Assume at first that $m = 0$ and consider set of *positive* roots of equation

$$w^{C*}(l) = \frac{\alpha^2\beta\varphi \cdot l}{(2\beta\varphi + \gamma l)^2} = C_u^C(l, 0, t) = \frac{2t}{3}\sqrt{\frac{l}{\pi}}.$$

Dividing both sides by $\sqrt{l}$ and substituting $x = \sqrt{l}$ we obtain the equivalent equation

$$3\alpha^2\beta\varphi\sqrt{\pi} \cdot x = 2t\left(2\beta\varphi + \gamma x^2\right)^2 \iff 8t\beta^2\varphi^2 - 3\alpha^2\beta\varphi\sqrt{\pi} \cdot x + 8t\beta\gamma\varphi x^2 + \gamma^2 x^4 = 0.$$

Sign of coefficients changes twice, hence, this equation has either 2, or 0 positive roots, due to Descartes' rule of signs. On the other hand, $w^{C*}(l^*) > C_u^C(l^*, 0, t)$ and $w^{C*}(l) < C_u^C(l, 0, t)$ for sufficiently large t, i.e., there is at least one positive root. It implies that these roots are $l_{\text{min}}(0,t)$ and $l_{\text{max}}(0,t)$, respectively, and $w^{C*}(l) - C_u^C(l, 0, t) \geq 0$ if and only if $l \in [l_{\text{min}}(0,t), l_{\text{max}}(0,t)]$. Moreover, it was proved that $l_{\text{max}}(0,t) > l^*$ and $l_{\text{min}}(0,t) < l^P$.

Now let $m > 0$, then $C_u^C(l, m) \equiv C_u^C(l, 0)$ for all $l \in [0, l^P]$ and $C_u^C(l, m) > C_u^C(l, 0)$ for all $l > l^M$ by Proposition 2. Let's show that equation $w^{C*}(l) = C_u^C(l, m, t)$ also has two positive roots $l_{\text{min}}(m,t)$ and $l_{\text{max}}(m,t)$, such that $w^{C*}(l) \geq C_u^C(l, m, t)$ if and only if $l \in [l_{\text{min}}(m,t), l_{\text{max}}(m,t)]$. Indeed, $C_u^C(l, m, t) \equiv C_u^C(l, 0, t)$ for all $l \leq l^P$, thus $l_{\text{min}}(m,t) \equiv l_{\text{min}}(0,t) \in (0, l^P)$. There is no roots in interval $(l_{\text{min}}(0,t), l_{\text{max}}(0,t))$, because $C_u^C(l, m, t) \leq C_u^C(l, 0, t) < w^{C*}(l)$. Therefore, there is a unique root of equation $w^{C*}(l) = C_u^C(l, m, t)$ on interval $(l_{\text{max}}(0,t), +\infty)$, because $w^{C*}(l)$ strictly decreases for all $l > l_{\max}(0,t) > l^*$, while $C_u^C(l,m,t)$ strictly increases on $(0,+\infty)$. This completes the proof of proposition.

Proof of Lemma 2

From budget constraint for the Home consumer (17)

$$q_0 = E - \int_0^{n_H} p_{HH}(i)q_{HH}(i)\mathrm{d}i - \int_{n_H}^{n_H+n_F} p_{FH}(i)q_{FH}(i)\mathrm{d}i,$$

substituting it into utility (1) we obtain the following FOC for Home:

$$\begin{aligned}\frac{\partial U}{\partial q_{HH}(i)} &= \alpha - \beta q_{HH}(i) - \gamma\left[\int_0^{n_H} q_{HH}(j)\mathrm{d}j + \int_{n_H}^{n_H+n_F} q_{FH}(j)\mathrm{d}j\right] - p_{HH}(i) = 0,\\ \frac{\partial U}{\partial q_{FH}(i)} &= \alpha - \beta q_{FH}(i) - \gamma\left[\int_0^{n_H} q_{HH}(j)\mathrm{d}j + \int_{n_H}^{n_H+n_F} q_{FH}(j)\mathrm{d}j\right] - p_{FH}(i) = 0,\end{aligned} \tag{27}$$

and after integrating:

$$\begin{aligned}\alpha n_H - \beta\int_0^{n_H} q_{HH}(i)\mathrm{d}i - \gamma n_H\left[\int_0^{n_H} q_{HH}(j)\mathrm{d}j + \int_{n_H}^{n_H+n_F} q_{FH}(j)\mathrm{d}j\right] - \int_0^{n_H} p_{HH}(i)\mathrm{d}i = 0,\\ \alpha n_F - \beta\int_0^{n_F} q_{FH}(i)\mathrm{d}i - \gamma n_F\left[\int_0^{n_H} q_{HH}(j)\mathrm{d}j + \int_{n_H}^{n_H+n_F} q_{FH}(j)\mathrm{d}j\right] - \int_0^{n_F} p_{FH}(i)\mathrm{d}i = 0.\end{aligned} \tag{28}$$

Let

$$P_H = \int_0^{n_H} p_{HH}(i)\mathrm{d}i + \int_0^{n_F} p_{FH}(i)\mathrm{d}i$$

denote price index, then summing two equation of (26) we obtain

$$\alpha n - (\beta+\gamma n)\left[\int_0^{n_H} q_{HH}(j)\mathrm{d}j + \int_{n_H}^{n_H+n_F} q_{FH}(j)\mathrm{d}j\right] - P_H = 0,$$

where $n = n_H + n_F$, or

$$\int_0^{n_H} q_{HH}(j)\mathrm{d}j + \int_{n_H}^{n_H+n_F} q_{FH}(j)\mathrm{d}j = \frac{\alpha n - P_H}{\beta+\gamma n}.$$

Substituting this into and (26), we obtain

$$\begin{aligned} q_{HH}(i) &= \frac{\alpha}{\beta+\gamma n} - \frac{1}{\beta}p_{HH}(i) + \frac{\gamma}{(\beta+\gamma n)\beta}\cdot P_H \\ q_{FH}(i) &= \frac{\alpha}{\beta+\gamma n} - \frac{1}{\beta}p_{FH}(i) + \frac{\gamma}{(\beta+\gamma n)\beta}\cdot P_H \end{aligned}.$$

Similarly

$$\begin{aligned} q_{FF}(i) &= \frac{\alpha}{\beta+\gamma n} - \frac{1}{\beta}p_{FF}(i) + \frac{\gamma}{(\beta+\gamma n)\beta}\cdot P_F \\ q_{HF}(i) &= \frac{\alpha}{\beta+\gamma n} - \frac{1}{\beta}p_{HF}(i) + \frac{\gamma}{(\beta+\gamma n)\beta}\cdot P_F \end{aligned}$$

where

$$P_F = \int_0^{n_H} p_{HF}(i)\mathrm{d}i + \int_0^{n_F} p_{FF}(i)\mathrm{d}i.$$

Now turn to firm's problems $I_H(i) \to$ max, $I_F(i) \to$ max, where

$$I_H(i) = \lambda L \cdot p_{HH}(i)\cdot q_{HH}(i) + (1-\lambda)L\cdot[p_{HF}(i)-\tau]\cdot q_{HF}(i)$$

$$I_F(i) = (1-\lambda)L \cdot p_{FF}(i)\cdot q_{FF}(i) + \lambda L\cdot[p_{FH}(i)-\tau]\cdot q_{FH}(i)$$

From the corresponding FOCs we obtain that equilibrium prices are

$$\begin{aligned} p^*_{HH}(i) \equiv p^*_{HH} &= \frac{\alpha\beta}{2(\beta+\gamma n)} + \frac{\gamma}{2(\beta+\gamma n)}\cdot P_H \\ p^*_{FH}(i) \equiv p^*_{FH} &= \frac{\alpha\beta}{2(\beta+\gamma n)} + \frac{\gamma}{2(\beta+\gamma n)}\cdot P_H + \frac{\tau}{2} \\ p^*_{FF}(i) \equiv p^*_{FF} &= \frac{\alpha\beta}{2(\beta+\gamma n)} + \frac{\gamma}{2(\beta+\gamma n)}\cdot P_F \\ p^*_{HF}(i) \equiv p^*_{HF} &= \frac{\alpha\beta}{2(\beta+\gamma n)} + \frac{\gamma}{2(\beta+\gamma n)}\cdot P_F + \frac{\tau}{2} \end{aligned}$$

On the other hand,

$$P_H = \int_0^{n_H} p_{HH}(i)\mathrm{d}i + \int_0^{n_F} p_{FH}(i)\mathrm{d}i,\ \ P_F = \int_0^{n_H} p_{HF}(i)\mathrm{d}i + \int_0^{n_F} p_{FF}(i)\mathrm{d}i$$

which imply

$$\begin{aligned} P_H &= \frac{\alpha\beta n + (\beta+\gamma n)\tau n_F}{2\beta+\gamma n} \\ P_F &= \frac{\alpha\beta n + (\beta+\gamma n)\tau n_H}{2\beta+\gamma n} \end{aligned}$$

The Home Consumer's surplus with respect to both domestic and imported differentiated goods is

$$CS_H = n_H \Delta_{HH} + n_F \Delta_{FH},$$

where Δ_{HH}, Δ_{FH} are consumer's surpluses with respect to specific domestic and imported varieties, i.e., areas of specific orthogonal triangles:

$$\Delta_{HH} = \frac{1}{2} q^*_{HH} \cdot \left(\frac{\alpha\beta + \gamma P_H}{\beta + \gamma n} - p^*_{HH} \right),$$
$$\Delta_{FH} = \frac{1}{2} q^*_{FH} \cdot \left(\frac{\alpha\beta + \gamma P_H}{\beta + \gamma n} - p^*_{FH} \right).$$

Substituting q^*_{HH}, q^*_{FH} , p^*_{HH}, p^*_{FH} and P_H we obtain after transformations

$$CS_H = \frac{\alpha^2 n}{2(\beta + \gamma n)} - \frac{\alpha}{\beta + \gamma n} \cdot [p^*_{HH} \cdot n_H + p^*_{FH} \cdot n_F] +$$

$$+ \frac{1}{2\beta} \cdot \left[(p^*_{HH})^2 \cdot n_H + (p^*_{FH})^2 \cdot n_F \right] - \frac{\gamma}{2\beta \cdot (\beta + \gamma n)} \cdot [p^*_{HH} \cdot n_H + p^*_{FH} \cdot n_F]^2$$

The trade profitability condition $p^*_{HF} > \tau,\ p^*_{FH} > \tau$ implies

$$p^*_{HH} = \frac{2\alpha\beta + \tau\gamma n_F}{2(2\beta + \gamma n)} > \frac{\tau}{2},\ p^*_{FF} = \frac{2\alpha\beta + \tau\gamma n_H}{2(2\beta + \gamma n)} > \frac{\tau}{2},$$

which is equivalent to

$$\tau < \min \left\{ \frac{2\alpha\beta}{2\beta + \gamma n_H}, \frac{2\alpha\beta}{2\beta + \gamma n_F} \right\} < \alpha.$$

Thus $\tau < \alpha$ is necessary condition for bilateral trade. On the other hand, max $\{n_H, n_H\} < n$, therefore

$$\min \left\{ \frac{2\alpha\beta}{2\beta + \gamma n_H}, \frac{2\alpha\beta}{2\beta + \gamma n_F} \right\} > \frac{2\alpha\beta}{2\beta + \gamma n}$$

and inequality is $\tau < \tau_{\text{trade}} = \dfrac{2\alpha\beta}{2\beta + \gamma n}$ is a sufficient condition for profitability of bilateral trade.

Proof of Proposition 7

Note that

$$w_H^{C*}(0) = \frac{\alpha^2 \beta\varphi L}{(2\beta\varphi + \gamma L)^2} \left(1 - \tau \cdot \frac{2\alpha\beta\varphi + \gamma L}{2\alpha\beta\varphi} \right)^2 > 0 = C_u^C(0)$$

for all $\tau < \tau_{\text{trade}} = \dfrac{2\alpha\beta\varphi}{2\beta\varphi + \gamma L}$. Moreover, substituting $\tau = 0$ into (19) we obtain

$$w_H^{C*}(\lambda) \equiv \frac{\alpha^2\beta\varphi L}{(2\beta\varphi + \gamma L)^2} = w_H^{C*}(1) \geq C_{uH}^C(1) > C_{uH}^C(\lambda)$$

for all $\lambda \in (0, 1)$, because $L < l_{\max}(m_H)$. Thus, for all sufficiently small $\tau < \tau^*$ inequality $w_H^{C*}(\lambda) > C_{uH}^C(\lambda)$ holds for all $\lambda \in (0, 1)$.

Let

$$\Delta(\lambda, \tau) = V_H(\lambda, \tau) - V_H(\lambda) = \left(w_H^{C*}(\lambda, \tau) + CS_H(\lambda, \tau)\right) - \left(w_H^{C*}(\lambda) + CS_H(\lambda)\right).$$

We are about to prove that $\Delta(\lambda, \tau) > 0$ for all $\lambda \in (0, 1)$ and sufficiently small $\tau > 0$. Note that $\Delta(1, \tau) = 0$ and

$$\Delta(0, \tau) = \frac{L\left(2\varphi L\beta\gamma(\alpha - 3\tau)(\alpha - \tau) + 6\varphi^2\beta^2(\alpha - \tau)^2 + L^2\gamma^2\tau^2\right)}{4\varphi\beta(2\varphi\beta + L\gamma)^2}.$$

Quadratic equation

$$2\varphi L\beta\gamma(\alpha - 3\tau)(\alpha - \tau) + 6\varphi^2\beta^2(\alpha - \tau)^2 + L^2\gamma^2\tau^2 = 0$$

has no real solutions with respect to τ, while $\Delta(0, 0) > 0$. It implies that $\Delta(0, \tau) > 0 = \Delta(1, \tau)$ for all τ. Now we are about to prove that $\Delta(\lambda, \tau)$ is decreasing function. Note that

$$\frac{\partial \Delta}{\partial \lambda}(\lambda, 0) = -\frac{\varphi L\alpha^2\beta(6\varphi\beta + L\gamma\lambda)}{2(2\varphi\beta + L\gamma\lambda)^3} < 0$$

for all $\lambda \in (0, 1)$. Thus, for sufficiently small $\tau < \tau^{**}$ inequality $\dfrac{\partial \Delta}{\partial \lambda}(\lambda, \tau) < 0$ holds for all λ.

Proof of Proposition 8

First assume that $G = 2$, i.e., there are two autarchic cities, H and F, with population $l_H = \lambda L$, $l_F = (1 - \lambda)L$, where $\lambda \in (0, 1)$, we obtain

$$\begin{aligned} n_H &= \tfrac{\beta}{\gamma}\left[(A \cdot L \cdot \lambda - 1) + \sqrt{(A \cdot L \cdot \lambda - 1)^2 + B \cdot L \cdot \lambda}\right], \\ n_F &= \tfrac{\beta}{\gamma}\left[(A \cdot L \cdot (1 - \lambda) - 1) + \sqrt{(A \cdot L \cdot (1 - \lambda) - 1)^2 + B \cdot L \cdot (1 - \lambda)}\right] \end{aligned} \tag{29}$$

while the mass of firms in joint city is

$$n^*(L) = \frac{\beta}{\gamma}\left[(A \cdot L - 1) + \sqrt{(A \cdot L - 1)^2 + B \cdot L}\right]$$

Note that inequality $n_H + n_F > n^*(L)$ is equivalent to

$$\begin{gathered}\sqrt{(A \cdot L \cdot \lambda - 1)^2 + B \cdot L \cdot \lambda} + \sqrt{(A \cdot L \cdot (1-\lambda) - 1)^2 + B \cdot L \cdot (1-\lambda)} > \\ > 1 + \sqrt{(A \cdot L - 1)^2 + B \cdot L}\end{gathered}$$

Both sides are positive, thus squaring this inequality, we obtain the equivalent one, which is, after collecting terms:

$$\begin{gathered}\sqrt{((A \cdot L \cdot \lambda - 1)^2 + B \cdot L \cdot \lambda) \cdot ((A \cdot L \cdot (1-\lambda) - 1)^2 + B \cdot L \cdot (1-\lambda))} > \\ > A^2 L^2 \cdot \lambda \cdot (1-\lambda) + \sqrt{(A \cdot L - 1)^2 + B \cdot L}\end{gathered}.$$

Squaring this inequality once again, we obtain

$$\begin{gathered}(A^2L^2 \cdot \lambda^2 + 1 + (B-2A)L \cdot \lambda) \cdot (A^2L^2 \cdot (1-\lambda)^2 + 1 + (B-2A)L \cdot (1-\lambda)) > \\ > A^4L^4 \cdot \lambda^2 \cdot (1-\lambda)^2 + (A^2L^2 + 1 + (B-2A)L) + 2A^2L^2\lambda \cdot (1-\lambda)\sqrt{A^2L^2 + 1 + (B-2A)L}\end{gathered},$$

which is equivalent to

$$A^2 L \cdot \left(\frac{B-2A}{2}\right) + 2\left(\frac{B-2A}{2}\right)^2 - A^2 > A^2\sqrt{A^2L^2 + 1 + (B-2A)L}$$

This inequality obviously holds for $A = 0$, thus we assume that $A \neq 0$. Recall that $A = \dfrac{\gamma - \alpha v}{2\beta f}$, $B = \dfrac{2\gamma}{\beta f}$, hence $\dfrac{B-2A}{2} = \dfrac{\gamma + \alpha v}{2\beta f} > A$, which transforms the previous inequality into

$$\begin{gathered}\left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \frac{\gamma + \alpha v}{2\beta f} L + 2\left(\frac{\gamma + \alpha v}{2\beta f}\right)^2 - \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 > \\ > \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \sqrt{\left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 L^2 + 1 + 2 \cdot \frac{\gamma + \alpha v}{2\beta f} L}\end{gathered}.$$

Also, it is sufficient to prove the last inequality, taking into account that $v > 0$. Indeed,

$$L.H.S. = \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \frac{\gamma + \alpha v}{2\beta f} L + 2\left(\frac{\gamma + \alpha v}{2\beta f}\right)^2 - \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 >$$
$$< \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \frac{\gamma + \alpha v}{2\beta f} L + 2\left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 - \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 = \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \left(\frac{\gamma + \alpha v}{2\beta f} L + 1\right) =$$
$$= \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \sqrt{\left(\frac{\gamma + \alpha v}{2\beta f} L + 1\right)^2} =$$
$$= \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \sqrt{1 + \left(\frac{\gamma + \alpha v}{2\beta f} L\right)^2 + 2 \cdot \frac{\gamma + \alpha v}{2\beta f} L} >$$
$$> \left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 \sqrt{\left(\frac{\gamma - \alpha v}{2\beta f}\right)^2 L^2 + 1 + 2 \cdot \frac{\gamma + \alpha v}{2\beta f} L} = R.H.S$$

Now the general case can be proved by induction

$$\sum_{g=1}^{G} n_g = \sum_{g=1}^{G-1} n_g + n_G > n^* \left(\sum_{g=1}^{G-1} l_g\right) + n_G > n^*(L).$$

Proof of Proposition 10

Due to Lemma 4 and Implicit Function Theorem we have to prove all statements for $\lambda = 1/2$ only.

i)

$$n_H^*(1/2) + n_F^*(1/2) - n^*(1) =$$
$$= \frac{\beta}{\gamma}\left[(A_\tau - A) \cdot L + \sqrt{(A_\tau \cdot L - 1)^2 + B \cdot L} - \sqrt{(A \cdot L - 1)^2 + B \cdot L}\right] =$$
$$= \frac{\beta}{\gamma}\left[(A_\tau - A) \cdot L + \frac{(A_\tau \cdot L - 1)^2 - (A \cdot L - 1)^2}{\sqrt{(A_\tau \cdot L - 1)^2 + B \cdot L} + \sqrt{(A \cdot L - 1)^2 + B \cdot L}}\right] =$$
$$= \frac{\beta}{\gamma}(A_\tau - A) \cdot L \left[1 + \frac{(A_\tau \cdot L - 1) + (A \cdot L - 1)}{\sqrt{(A_\tau \cdot L - 1)^2 + B \cdot L} + \sqrt{(A \cdot L - 1)^2 + B \cdot L}}\right] =$$
$$= \frac{\tau v}{4\gamma f} \cdot L \cdot \frac{\sqrt{(A_\tau \cdot L - 1)^2 + B \cdot L} + (A_\tau \cdot L - 1) + \sqrt{(A \cdot L - 1)^2 + B \cdot L} + (A \cdot L - 1)}{\sqrt{(A_\tau \cdot L - 1)^2 + B \cdot L} + \sqrt{(A \cdot L - 1)^2 + B \cdot L}} > 0$$

because $\sqrt{(A_\tau \cdot L - 1)^2 + B \cdot L} > |A_\tau \cdot L - 1|$, $\sqrt{(A \cdot L - 1)^2 + B \cdot L} > |A \cdot L - 1|$.

ii) Direct calculations show that

$$\frac{\partial n_H^*}{\partial \lambda}(\lambda_0, v_0) = -\frac{\partial n_F^*}{\partial \lambda}(\lambda_0, v_0) = \frac{L}{f} > 0$$

The rest of inequalities for ii), as well as for iii)-v) are straightforward from (23).