Games on Networks

Yves Zenou (Stockholm University and IFN)

GAMES ON NETWORKS

Take the network as given and study the impact of network structure on outcomes.

GAMES WITH STRATEGIC COMPLEMENT

VS

GAMES WITH STRATEGIC SUBSTITUTES

Defining Strategic Complements and Substitutes Players in N have action spaces A_i.

Let $A = A_1 \times \cdots A_n$.

Player i's payoff function is denoted $u_i : A \times G(N) \to \mathbb{R}$.

Equilibrium: Pure strategy Nash equilibrium: a profile of actions $\mathbf{a} \in A = A_1 \times \cdots A_n$, such that

$$u_i(a_i, \mathbf{a}_{-i}, \mathbf{g}) \geq u_i(a_i', \mathbf{a}_{-i}, \mathbf{g})$$

for all $a_i' \in A_i$.

Take A_i (the action space) to be a complete lattice with an associated partial order $\geq_i$, for each i.

Then it is easy to see that A is also complete lattice,

if we define $\mathbf{a} \geq \mathbf{a}'$ if and only if $a_i \geq a_i'$ for every i,

and where for any $S \subset A$ we define $\inf(S) = (\inf_i\{a_i : \mathbf{a} \in S\})_i$ and $\sup(S) = (\sup_i\{a_i : \mathbf{a} \in S\})_i$.

A game exhibits *strategic complements* if it exhibits *increasing differences*; that is, for all i, $a_i \geq_i a_i'$ and $a_{-i} \geq_{-i} a'_{-i}$:

$$u_i(a_i, a_{-i}, \mathbf{g}) - u_i(a_i', a_{-i}, \mathbf{g}) \geq u_i(a_i, a'_{-i}, \mathbf{g}) - u_i(a_i', a'_{-i}, \mathbf{g}).$$

or in other words if the difference $u_i(a_i, ., \mathbf{g}) - u_i(a_i', ., \mathbf{g})$ is an **increasing** function.

A game exhibits *strategic substitutes* if it exhibits *decreasing differences*; that is, for all i, j, with $i \neq j$, $a_i \geq_i a'_i$ and $a_{-i} \geq_{-i} a'_{-i}$:

$$u_i(a_i, a_{-i}, \mathbf{g}) - u_i(a'_i, a_{-i}, \mathbf{g}) \leq u_i(a_i, a'_{-i}, \mathbf{g}) - u_i(a'_i, a'_{-i}, \mathbf{g}).$$

or in other words if the difference $u_i(a_i, ., \mathbf{g}) - u_i(a'_i, ., \mathbf{g})$ is an **decreasing** function.

These notions are said to apply strictly if the inequalities above are strict whenever $a_i >_i a_i'$ and $a_{-i} \geq_{-i} a_{-i}'$ with $a_j >_j a_j'$ for $j \in N_i(\mathbf{g})$.

Examples

A given player's payoff depends on other players' behaviors, but only on those to whom the player is (directly) linked in the network.

For any i, a_i, and $\mathbf{g}$:

$$u_i(a_i, \mathbf{a}_{-i}, \mathbf{g}) = u_i(a_i, \mathbf{a}'_{-i}, \mathbf{g}),$$

whenever $\mathbf{a}_j = \mathbf{a}'_j$ for all $j \in N_i(\mathbf{g})$.

Example 1 *The Majority Game (Game with strategic complements)*

Players' action spaces are $A_i = \{0, 1\}$.

A player can choose to either do something or not to, for instance, buying a product, attending a party, and so forth.

Payoff to a player from taking action 1 compared to action 0 depends on the number of neighbors who choose action 1, so that

$$
\begin{aligned}
& sign\left(u_i(1, a_{N_i(\mathbf{g})}) - u_i(0, a_{N_i(\mathbf{g})})\right) \\
= \; & sign\left(\sum_{j \in N_i(\mathbf{g})} a_j - \sum_{j \in N_i(\mathbf{g})} (1 - a_j)\right).
\end{aligned}
$$

If more than one half of i's neighbors choose action 1, then it is best for player i to choose 1, and if fewer than one half of i's neighbors choose action 1 then it is best for player i to choose action 0.

Multiple equilibria

For example, all players taking action 0 (or 1) is an equilibrium.

Another equilibrium

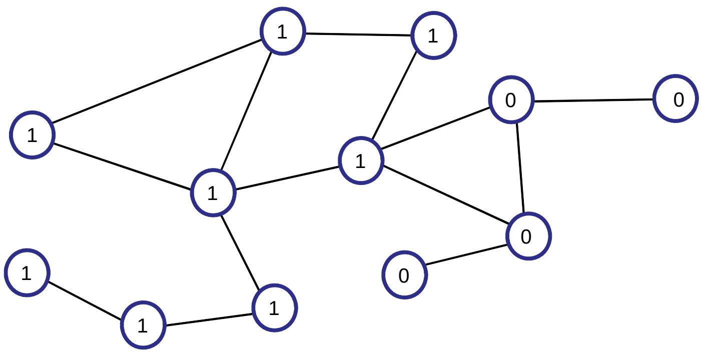


Figure 1: An equilibrium in the majority game

Example 2 *"Best-Shot" Public Goods Games (Game with strategic substitutes)*

For instance, the action might be learning how to do something, where that information is easily communicated; or buying a book or other product that is easily lent from one player to another.

Taking the action 1 is costly and a player would prefer that a neighbor take the action than having to do it himself or herself; but, taking the action and paying the cost is better than having nobody take the action.

$$u_i(\mathbf{a}, \mathbf{g}) = \begin{cases} 1 - c & \text{if } a_i = 1, \\ 1 & \text{if } a_i = 0,\ a_j = 1 \text{ for some } j \in N_i(\mathbf{g}) \\ 0 & \text{if } a_i = 0,\ a_j = 0 \text{ for all } j \in N_i(\mathbf{g}), \end{cases}$$

where $1 > c > 0$.

There are many possible equilibria in the best-shot public goods game

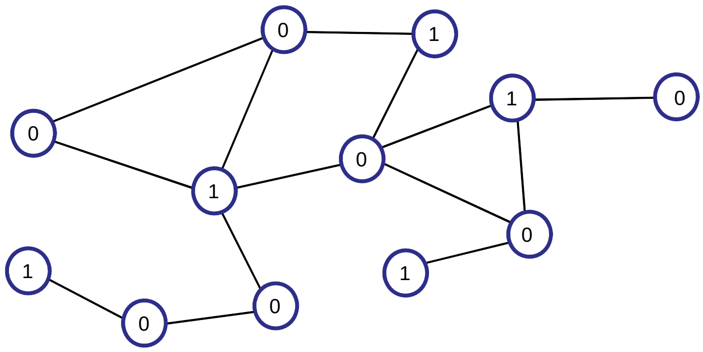


Figure 2: An equilibrium in the best-shot public good game

Definitions

An *independent set* relative to a graph $(N, \mathbf{g})$ is a subset of nodes $A \subset N$ for which no two nodes are adjacent (i.e., linked).

A is a maximal independent set if there does not exist another independent, set $A' \neq A$, such that $A \subset A' \subset N$.

A *dominating set* relative a graph $(N, \mathbf{g})$ is a subset of nodes $A \subset N$ such that every node in A is connected to every other node in A via a path that involves only nodes in A, and every node not in A is linked to at least one member of A.

For example, the central node in a **star** forms a **dominating set** and also a **maximal independent set**, while each peripheral node is an independent set and the set of all peripheral nodes is a maximal independent set.

Any set including the central node and some periperhal nodes is a dominating set, but not an independent set.

The equilibria in the "Best-Shot" Public Goods Game correspond exactly to having the set of players who choose action 1 form a **maximal independent set of nodes in the network** (Bramoullé and Kranton (2007)) that is, a maximal set of nodes that have no links to each other in the network.

Nobody wants to deviate from their Nash equilibrium actions.

The central player who chooses action 1. His/her utility is $1 - c$. Since all his/her neighbors choose action 0, deviating by choosing action 0 would give him/her a utility of $0 < 1 - c$.

Similarly, for each player who chooses action 0, his/her utility is 1 since at least one of his/her neighbors choose action 1. Choosing action 1 would give him/her $1 - c < 1$.

For **smooth functions**, supermodularity/increasing differences admit a convenient test.

Lemma: If $u_i(.,.,\mathbf{g})$ is twice continuously differentiable, increasing differences is equivalent to

$$\frac{\partial^2 u_i(a_i, a_{-i}, \mathbf{g})}{\partial a_i \partial a_{-i}} \geq 0$$

for all a_i and a_{-i}.

Similar result for decreasing differences which is equivalent to

$$\frac{\partial^2 u_i(a_i, a_{-i}, \mathbf{g})}{\partial a_i \partial a_{-i}} \leq 0$$

Proof: *Increasing differences* mean ($a_i = a_i + \varepsilon_i$; $a'_i = a_i$ and $a'_{-i} = a_{-i} + \varepsilon_{-i}$; $a'_{-i} = a_{-i}$)

$$
\begin{aligned}
& u_i(a_i + \varepsilon_i, a_{-i} + \varepsilon_{-i}, \mathbf{g}) - u_i(a_i, a_{-i} + \varepsilon_{-i}, \mathbf{g}) \\
\geq\ & u_i(a_i + \varepsilon_i, a_{-i}, \mathbf{g}) - u_i(a_i, a_{-i}, \mathbf{g}).
\end{aligned}
$$

Divide the two sides of the inequality by the positive quantity ε_i and let ε_i tends to 0, we obtain:

$$
\begin{aligned}
& \lim_{\varepsilon_i \to 0} \frac{u_i(a_i + \varepsilon_i, a_{-i} + \varepsilon_{-i}, \mathbf{g}) - u_i(a_i, a_{-i} + \varepsilon_{-i}, \mathbf{g})}{\varepsilon_i} \\
\geq\ & \lim_{\varepsilon_i \to 0} \frac{u_i(a_i + \varepsilon_i, a_{-i}, \mathbf{g}) - u_i(a_i, a_{-i}, \mathbf{g})}{\varepsilon_i}
\end{aligned}
$$

$$
\Leftrightarrow \frac{\partial u_i(a_i, a_{-i} + \varepsilon_{-i}, \mathbf{g})}{\partial a_i} \geq \frac{\partial u_i(a_i, a_{-i}, \mathbf{g})}{\partial a_i}
$$

$$
\Leftrightarrow \frac{\partial u_i(a_i, a_{-i} + \varepsilon_{-i}, \mathbf{g})}{\partial a_i} - \frac{\partial u_i(a_i, a_{-i}, \mathbf{g})}{\partial a_i} \geq 0
$$

Divide the two sides of the inequality by the positive quantity ε_{-i} and let ε_{-i} tends to 0, we obtain:

$$\lim_{\varepsilon_{-i}\to 0}\left[\frac{\frac{\partial u_i(a_i,a_{-i}+\varepsilon_{-i},\mathbf{g})}{\partial a_i}}{\varepsilon_{-i}}-\frac{\frac{\partial u_i(a_i,a_{-i},\mathbf{g})}{\partial a_i}}{\varepsilon_{-i}}\right]\geq 0$$

$$\Leftrightarrow \frac{\partial^2 u_i(a_i, a_{-i}, \mathbf{g})}{\partial a_i \partial a_{-i}} \geq 0$$

Definitions

Let $\geq_i$ be a partial order on a (nonempty) set A_i (so $\geq_i$ is reflexive, transitive and antisymmetric).

$(A_i, \geq_i)$ is a **lattice** if any two elements a_i and a_i' have a **least upper bound** (supremum for i, $\mathbf{sup}_i$, such that $\mathsf{sup}_i(a_i, a_i') \geq a_i$ and $\mathsf{sup}_i(a_i, a_i') \geq a_i'$),

and a **greatest lower bound** (infimum for i, such that $\inf_i(a_i, a_i') \leq a_i$ and $\inf_i(a_i, a_i') \leq a_i'$), in the set.

A *lattice* $(A_i, \geq_i)$ is *complete* if every nonempty subset of A_i has a supremum and an infimum in A_i.

Take A_i (the action space) to be a complete lattice with an associated partial order $\geq_i$, for each i.

Then it is easy to see that A is also complete lattice,

if we define $\mathbf{a} \geq \mathbf{a}'$ if and only if $a_i \geq a_i'$ for every i,

and where for any $S \subset A$ we define $\inf(S) = (\inf_i\{a_i : \mathbf{a} \in S\})_i$ and $\sup(S) = (\sup_i\{a_i : \mathbf{a} \in S\})_i$.

A nonempty set of best responses $BR_i(a_{-i})$ is a **closed sublattice** of the complete lattice A_i if

$$\sup_i(BR_i(a_{-i})) \in BR_i(a_{-i})$$

and

$$\inf_i(BR_i(a_{-i})) \in BR_i(a_{-i}))$$

Existence of Equilibrium: Games of Strategic Complements Games of strategic complements are well-behaved: Not only do equilibria generally exist, but they form a lattice so that they are well-ordered and there are easy algorithms for finding the maximal and minimal equilibria.

Theorem 1 *Consider a game of strategic complements such that:*

- *for every player i, and specification of strategies of the other players, $a_{-i} \in A_{-i}$, player i has a nonempty set of best responses $BR_i(a_{-i})$ that is a closed sublattice of the complete lattice A_i, and*

- *for every player i, if $a'_{-i} \geq a_{-i}$, then $\sup_i BR_i(a'_{-i}) \geq_i \sup_i BR_i(a_{-i})$ and $\inf_i BR_i(a'_{-i}) \geq_i \inf_i BR_i(a_{-i})$.*

An equilibrium exists and the set of equilibria form a (nonempty) complete lattice.

In games of strategic complements such that the set of actions is finite, or compact and payoffs are continuous, the conditions of the theorem apply and there exists an equilibrium.

Note that the equilibria exist in pure strategies, directly in terms of the actions A without requiring any additional randomizations.

The same is not true games of strategic substitutes.

Finding maximal and minimal equilibria in a game of strategic complements is then quite easy.

Let us describe an algorithm for the case where A is finite.

Begin with all players playing the maximal action $a^0 = \overline{a}$. Let $a_i^1 = \mathrm{sup}_i(BR_i(a_{-i}^0))$ for each i and, iteratively, let $a_i^k = \mathrm{sup}_i(BR_i(a_{-i}^{k-1}))$.

It follows that a point such that $a^k = a^{k-1}$ is the maximal equilibrium point, and given the finite set of strategies this must occur in a finite number of iterations.

Analogously, starting from the minimal action and iterating upward, one can find the minimal equilibrium point.

This also means that dynamics that iterate on best response dynamics will generally converge to equilibrium points in such games (e.g., see Milgrom and Roberts, 1990).

Games with strategic substitutes

Moving beyond games of strategic complements, existence of equilibria and the structure of the set are no longer so nicely behaved.

Existence of equilibria can be guaranteed along standard lines: for instance equilibria exist if A_i is a nonempty, compact, and convex subset of a Euclidean space and u_i is continuous and quasi-concave for every i.

This covers the canonical case where A_i are the mixed strategies associated with an underlying finite set of pure actions and u_i is the expected payoff and hence quasi-concave.

Nonetheless, this means that pure strategy equilibria may not exist unless the game has some specific structure.

In addition, with the lack of lattice structure, best responses are no longer so nicely ordered and equilibria in many network games can be more difficult to find.

Some games of strategic substitutes on networks still have many important applications and are tractable in some cases.

For example, consider the best-shot public goods game discussed above.

As we showed above, best-shot public goods games on a network always have pure strategy equilibria, and in fact those equilibria are the situations where the players who take action 1 form a maximal independent set.

Finding all of the maximal independent sets is computationally intensive, but finding one such set is easy.

Here is an algorithm that finds an equilibrium.

At a given step k, the algorithm lists a set of the providers of the public good (the independent set of nodes), P_k,

and a set of non-providers of the public good (who will not be in the eventual maximal independent set of nodes), NP_k, where the eventual maximal independent set will be the final P_k.

In terms of finding an equilibrium to the best-shot game, the final P_k is the list of players who take action 1, and the final NP_k is the set of players who take action 0.

Step 1: Pick some node i and let $P_1 = \{i\}$ and $NP_1 = N_i(\mathbf{g})$.

Step k: Iterate by picking one of the players j who is not yet assigned to sets P_{k-1} or NP_{k-1}. Let $P_k = P_{k-1} \cup \{j\}$ and $NP_k = NP_{k-1} \cup N_j(\mathbf{g})$.

End: Stop when $P_k \cup NP_k = N$.

More generally, one might ask the question of whether it is possible to find the “best” equilibrium in the best-shot game.

Given that in every equilibrium all players get a payoff of either 1 or $1-c$, minimizing the number of players who pay the cost c would be one metric via which to rank equilibria.

As discussed by Dall'Asta, Pin and Ramezanpour (2011), finding such equilibria can be difficult but finding them (approximately) through an intuitive class of mechanisms that tradeoff accuracy against speed is possible.

There are other games of strategic substitutes where at least some equilibria are also easy to find.

Example 3 *A "Weakest-Link" Public Goods Game*

Here each player chooses some level of public good contribution (so $A_i \subset \mathbb{R}_+$) and the payoff to a player is the *minimum* action taken by any player in his or her neighborhood (in contrast to the maximum, as in the best-shot game).

We have:

$$u_i(a_i, a_{N_i(\mathbf{g})}) = \min_{j \in N_i(\mathbf{g}) \cup \{i\}} \left\{ a_j \right\} - c(a_i)$$

where c is an increasing, convex and differentiable cost function.

In the case where there is a smallest a^* such that $c'(a^*) \geq 1$, and each player has at least one neighbor in the network g,

any profile of actions where every player chooses the same contribution $a_i = a^*$ is an equilibrium of this game.

In a network in which every player has at least one neighbor, everyone playing $a_i = 0$ is also an equilibrium, and so the game will have multiple equilibria when it is non-degenerate.

Continuous Actions, Quadratic Payoffs,

and Strategic Complementarities

Who’s Who in Networks. Wanted: the Key Player

Coralio Ballester, Antoni Calvó-Armengol and Yves Zenou

Econometrica 2006

Peer effects: where do we stand?

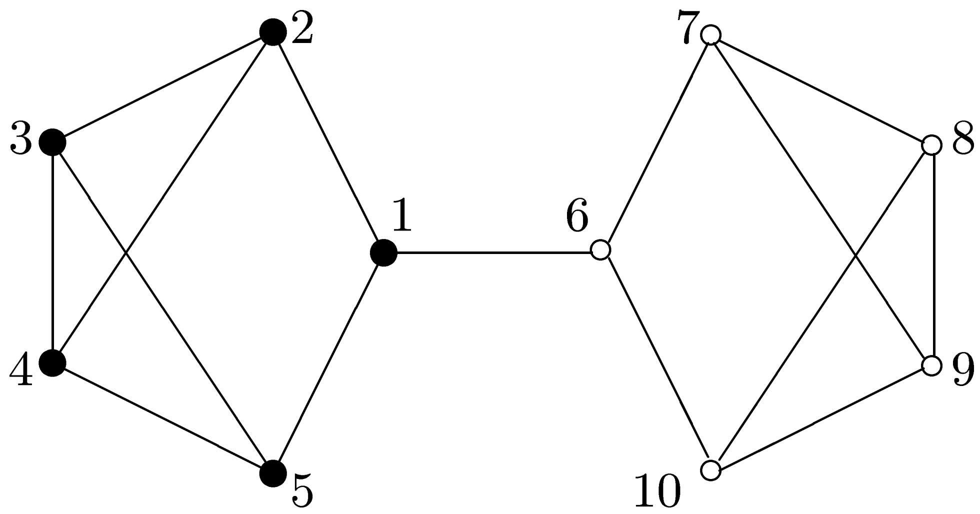
The influence of group outcome on members' behavior is documented in ethnographic and empirical studies (drop-outs, crime, etc.).

Theoretical models of peer effects typically assume that individual payoffs depend on average group outcomes, and derive consequences.

But, the generative mechanisms of peer effects remain a black-box...

A generative mechanism: labor market networks

Drop-out decisions vary with network location!



Agent	1	2	6	7
Drop-out Rate	0.47	0.42	0.91	0.93

What do we do?

We keep track of all the bilateral influences in the group $\rightarrow$ *network*.

We relate individual outcomes to network location $\rightarrow$ *who's who*.

We identify optimal targets in the network $\rightarrow$ *key players*.

We discuss three applications: crime, RandD, conformism.

A game with linear-quadratic payoffs

Player i selects an effort $x_i \geq 0$ and gets:

$$u_i(\mathbf{x}) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j .$$

We get a matrix of cross effects:

$$\mathbf{\Sigma} = \left[\frac{\partial^2 u_i}{\partial x_i \partial x_j} \right] = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix} .$$

Let $\underline{\sigma} = \min\{\sigma_{ij}\}$ and $\overline{\sigma} = \max\ \{\sigma_{ij}\}$.

Assumptions on the cross effects

Net of cross effects, players are identical: $\sigma_{ii} = \sigma$ and $\alpha_i = \alpha$.

Strategic complementarity/substitutability allowed: σ_{ij} unrestricted.

Own-concavity steeper than cross marginal returns: $\sigma < \min\{\underline{\sigma}, 0\}$.

A positive slope at $\mathbf{0}$: $\alpha > 0$.

When $\sigma_{ij} > 0$, an increase in the effort x_j of agent j creates an incentive for i to increase his level of activity x_i. We then talk of *strategic complementarity* in efforts.

When $\sigma_{ij} < 0$, instead, an extra effort from j triggers a downards shift in i's effort in response. We say that efforts are *strategic substitutes*.

The decomposition of cross effects

$$\mathbf{\Sigma} = -\,\beta \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{own-concavity} - \;\gamma \underbrace{\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}}_{global-susbtitutability} + \lambda \underbrace{\begin{bmatrix} 0 & & g_{ij} \\ & \ddots & \\ g_{ij} & & 0 \end{bmatrix}}_{local-complementarity},$$

where:

$\gamma = -\min\{\underline{\sigma}, 0\} \geq 0$, $\lambda = \overline{\sigma} + \gamma \geq 0$, and $\beta = -\sigma - \gamma$.

$g_{ij} = (\sigma_{ij} + \gamma)/\lambda$, with $0 \leq g_{ij} \leq 1$.

The General Model

- We can decompose bilateral influences like

$$\Sigma = \underbrace{-\beta \mathbf{I}}_{\text{Net Self-Substitutability}} \underbrace{-\gamma \mathbf{U}}_{\text{Global Substitutability}} \underbrace{+\lambda \mathbf{G}}_{\text{Local Complementarity}}$$

where G represents a network of local complementarities, $0 \le g_{ij} \le 1$

$\mathbf{I}$ is the $n-$square identity matrix and $\mathbf{U}$ the $n-$square matrix of ones.

$$\mathbf{\Sigma} = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G}$$

with $\beta > 0$, $\gamma \geq 0$ and $\lambda > 0$.

The pattern of bilateral influences results from the combination of an idiosyncratic effect, a global interaction effect, and a local interdependence component.

The idiosyncratic effect reflects (part of) the concavity of the payoff function in own efforts.

The global interaction effect is uniform across all players (matrix $\mathbf{U}$) and corresponds to a substitutability effect across all pairs of players with value $-\gamma \leq 0$.

The local interaction component captures the (relative) strategic complementarity in efforts that varies across pairs of players, with maximal strength λ and population pattern reflected by $\mathbf{G}$.

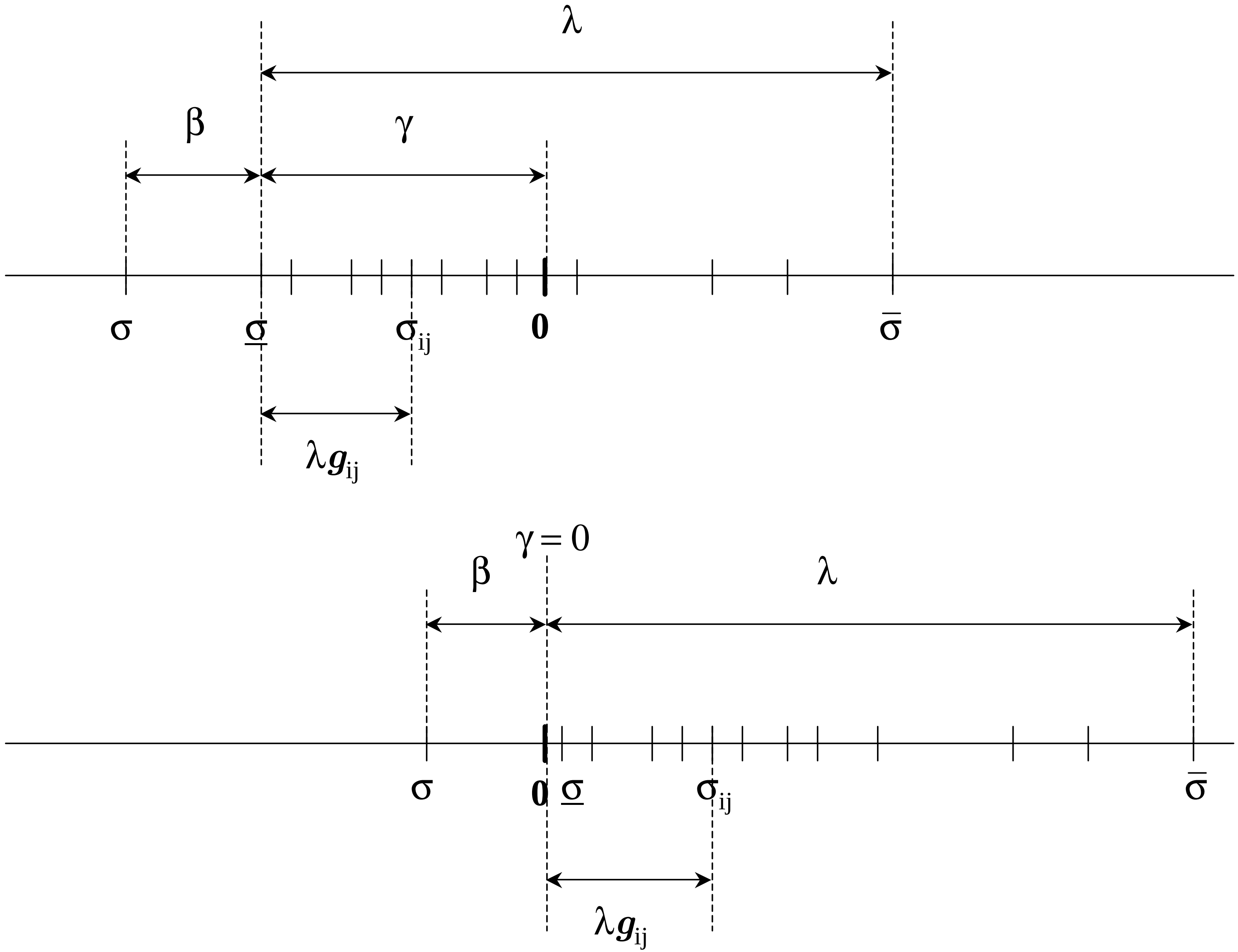
The decomposition is depicted in Figure 1.

This is just a centralization (β and λ are defined with respect to γ)

followed by a normalization (the g_{ij}s are in $[0,1]$) of the cross effects.

The figure in the upper panel corresponds to σ_{ij} of either sign (the case $\sigma_{ij} \leq 0$, for all $i \neq j$ is similar)

while the figure in the lower panel corresponds to $\sigma_{ij} \geq 0$, for all $i \neq j$ (recall that we assume $\sigma < 0$).


λ
β
γ
σ
σ̲
σ_ij
0
σ̄
λg_ij
γ = 0
β
λ
σ
0
σ̲
σ_ij
σ̄
λg_ij

The General Model

- Three players

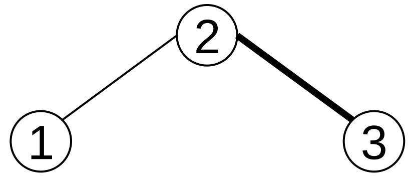
$$u_1(x_1,x_2,x_3) = x_1 - 3x_1^2 + \frac{1}{2}x_1x_2 - x_1x_3$$

$$u_2(x_1,x_2,x_3) = x_2 - 3x_2^2 + \frac{1}{2}x_1x_2 + x_2x_3$$

$$u_3(x_1,x_2,x_3) = x_3 - 3x_3^2 - x_1x_2 + x_1x_3$$

$$\Sigma = \begin{pmatrix} -6 & 1/2 & -1 \\ 1/2 & -6 & 1 \\ -1 & 1 & -6 \end{pmatrix}$$

$$\Sigma = \underbrace{-5}_{-\beta}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{-1}_{-\gamma}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \underbrace{+2}_{+\lambda}\begin{pmatrix} 0 & 3/4 & 0 \\ 3/4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



Explanation of this example:

$$\Sigma = \begin{pmatrix} -6 & 1/2 & -1 \\ 1/2 & -6 & 1 \\ -1 & 1 & -6 \end{pmatrix}$$

$$\underline{\sigma} = \min\{\sigma_{ij}\} = -1 \qquad \text{and} \qquad \overline{\sigma} = \max\{\sigma_{ij}\} = 1$$

OBS: $\underline{\sigma}$ and $\overline{\sigma}$ do not include $\sigma_{ii} = \sigma$.

$$\sigma_{ii} = \sigma = -6$$

$$\gamma = -\min\{\underline{\sigma}, 0\} = -\min\{-1, 0\} = 1$$

$$\lambda = \overline{\sigma} + \gamma = 1 + \gamma = 2$$

$$g_{ij} = \frac{\sigma_{ij} + \gamma}{\lambda} \text{ but } 0 \leq g_{ij} \leq 1$$

Thus

$$\mathbf{G} = \begin{pmatrix} \max\left\{\frac{-6+1}{2}, 0\right\} = 0 & \frac{1/2+1}{2} = \frac{3}{4} & \frac{-1+1}{2} = 0 \\ \frac{1/2+1}{2} = \frac{3}{4} & \max\left\{\frac{-6+1}{2}, 0\right\} = 0 & \frac{1+1}{2} = 1 \\ \frac{-1+1}{2} = 0 & \frac{1+1}{2} = 1 & \max\left\{\frac{-6+1}{2}, 0\right\} = 0 \end{pmatrix}$$

As a result

$$\begin{aligned} \mathbf{\Sigma} &= -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G} \\ &= -5 \times \mathbf{I} - 1 \times \mathbf{U} + 2 \times \mathbf{G} \end{aligned}$$

A new expression for payoffs

$$u_i(x_1, ..., x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j.$$

In our example, we have:

$$\alpha = 1 \; , \; \gamma = 1 \; , \; \lambda = 2 \; , \; \beta = 5$$

Since for all $i = 1, 2, 3$

$$u_i = \alpha x_i - \frac{1}{2}(\beta - \gamma)\, x_i^2 - \gamma \sum_{j=1}^{3} x_i x_j + \lambda \sum_{j=1}^{3} g_{ij} x_i x_j$$

we have:

$$\begin{aligned} u_1 &= \alpha x_1 - \frac{1}{2}(\beta - \gamma)\, x_1^2 - \gamma \sum_{j=1}^{3} x_1 x_j + \lambda \sum_{j=1}^{3} g_{1j} x_1 x_j \\ &= x_1 - \frac{1}{2} 4\, x_1^2 - x_1 x_1 - x_1 x_2 - x_1 x_3 + 2\frac{3}{4} x_1 x_2 \\ &= x_1 - 3x_1^2 + \frac{1}{2} x_1 x_2 - x_1 x_3 \end{aligned}$$

Similarly

$$u_2 = x_2 - 3x_2^2 + \frac{1}{2} x_1 x_2 + x_2 x_3$$

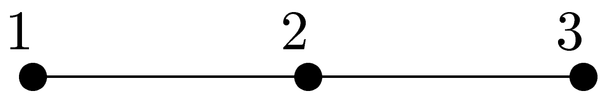
$$u_3 = x_3 - 3x_3^2 - x_1 x_2 + x_1 x_3$$

The network of local complementarities

$\mathbf{G} = [g_{ij}]$ is the adjacency matrix of a network $\mathbf{g}$.

When $\mathbf{\Sigma}$ is symmetric, $\mathbf{g}$ is un-directed.

When $\mathbf{\Sigma}$ is symmetric, $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ and $\underline{\sigma} \leq 0$, $\mathbf{G}$ is a symmetric $(0, 1)-$matrix and $\mathbf{g}$ a graph:



The network Bonacich centrality To each network $\mathbf{g}$, we associate its adjacency matrix $\mathbf{G} = [g_{ij}]$.

Symmetric zero-diagonal square matrix that keeps track of the direct connections in $\mathbf{g}$.

The kth power $\mathbf{G}^k = \mathbf{G}^{(k \ \underset{\dots}{times})}\mathbf{G}$ of the adjacency matrix $\mathbf{G}$ keeps track of indirect connections in $\mathbf{g}$.

The coefficient $g_{ij}^{[k]}$ in the (i, j) cell of $\mathbf{G}^k$ gives the number of paths of length k in $\mathbf{g}$ between i and j.

Example Network g with three individuals (star)

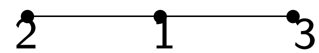


Figure 1

Adjacency matrix :

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{G}^{2k} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^{k-1} & 2^{k-1} \\ 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad \text{and} \quad \mathbf{G}^{2k+1} = \begin{bmatrix} 0 & 2^k & 2^k \\ 2^k & 0 & 0 \\ 2^k & 0 & 0 \end{bmatrix}, k \geq 1$$

$$\mathbf{G}^3 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$\mathbf{G}^3$: two paths of length three between 1 and 2: $12 \longrightarrow 21 \longrightarrow 12$ and $12 \longrightarrow 23 \longrightarrow 32$.

no path of length three from i to i

For all integer k, define:

$$b_i^k(\mathbf{g}) = \sum_{j=1}^{n} g_{ij}^{[k]}$$

This is the sum of all paths of length k in g starting from i.

Next, let $\phi \geq 0$, and define:

$$b_i(\mathbf{g},\phi) = \sum_{k=0}^{+\infty} \phi^k b_i^k(\mathbf{g})$$

This is the sum of all paths in g starting from i, where paths of length k are weighted by the geometrically decaying factor ϕ^k.

For ϕ small enough, this infinite sum takes on a finite value.

$$\mathbf{b}(\mathbf{g}, \phi) = \sum_{k=0}^{+\infty} \phi^k \mathbf{G}^k \cdot \mathbf{1} = [\mathbf{I} - \phi \mathbf{G}]^{-1} \cdot \mathbf{1}, \tag{2}$$

where $\mathbf{1}$ is the vector of ones.

$\mathbf{b}(\mathbf{g}, \phi)$ Bonacich network centrality of parameter ϕ in $\mathbf{g}$.

$b_i(\mathbf{g},\phi)$ as the Bonacich centrality of agent i in $\mathbf{g}$.

To each agent, it associates a value that counts the *total* number of direct and indirect (weighted) paths in the network stemming from this agent.

Example Consider the network $\mathbf{g}$ in Figure 1.

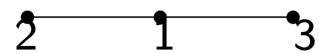


Figure 1

When ϕ is small enough, the vector of Bonacich network centralities is:

$$\mathbf{b}(\mathbf{g}, \phi) = \begin{bmatrix} b_1(\mathbf{g}, \phi) \\ b_2(\mathbf{g}, \phi) \\ b_3(g, \phi) \end{bmatrix} = \frac{1}{1 - 2\phi^2} \begin{bmatrix} 1 + 2\phi \\ 1 + \phi \\ 1 + \phi \end{bmatrix}.$$

The Bonacich centrality of node i is $b_i(\mathbf{g}, a) = \sum_{j=1}^{n} m_{ij}(\mathbf{g}, a)$, and counts the *total* number of paths in $\mathbf{g}$ starting from i.

It is the sum of all loops starting from i and ending at i, and all outer paths that connect i to every other player $j \neq i$:

$$b_i(\mathbf{g}, a) = \underbrace{m_{ii}(\mathbf{g}, a)}_{self-loops} + \sum_{j \neq i} \underbrace{m_{ij}(\mathbf{g}, a)}_{out-paths}.$$

Note that, by definition, $m_{ii}(\mathbf{g}, a) \geq 1$, and thus $b_i(\mathbf{g}, a) \geq 1$.

Nash equilibrium and Bonacich centrality

The largest eigenvalue $\mu_1(\mathbf{G})$ of $\mathbf{G}$, also called index of $\mathbf{g}$.

Theorem. $[\beta\mathbf{I} - \lambda\mathbf{G}]^{-1}$ is well-defined and non-negative iff $\beta > \lambda\mu_1(\mathbf{G})$. Then, $\mathbf{\Sigma}$ has a unique Nash equilibrium, interior:

$$\mathbf{x}^*(\mathbf{\Sigma}) = \frac{\alpha}{\beta + \gamma b(\mathbf{g}, \lambda/\beta)}\mathbf{b}(\mathbf{g}, \lambda/\beta)$$

Equilibrium outcomes are proportional to node centrality.

Nash equilibrium and Bonacich centrality (contd.)

Let $\mathbf{G}$ be a $(0,1)-$ matrix and $g = \sum_{i,j} g_{ij}$ twice number of edges.

Corollary. If $\beta > \lambda\sqrt{g+n-1}$, then Nash is Bonacich.

Remark. Suppose that $\alpha_i > 0$ differ across players. Then, Nash is a *weighted* Bonacich:

$$\mathbf{b}_\alpha(\mathbf{g}, \lambda^*) = [\mathbf{I}-\lambda^*\mathbf{G}]^{-1} \cdot \alpha$$

Best-reply functions

$$BR_i(\mathbf{y}_{-i}) = \phi \sum_{j=1}^{n} g_{ij} y_j + \sum_{m=1}^{M} \beta_m x_i^m - pf + \eta_k + \varepsilon_i$$

$$\underbrace{Alice}_{y_A \uparrow \Delta} \quad \rightarrow \quad \underbrace{Bob}_{y_B \uparrow \phi \Delta} \quad \rightarrow \quad \underbrace{Charlie}_{y_C \uparrow \phi^2 \Delta}$$

- Direct complementarities induce indirect complementarities of all possible order.
- There is a discount of distance ϕ^{distance}.
- This means that ϕ cannot be too large.

Spectral radius: Why?

Diagonalize $\mathbf{G}$:

$$\mathbf{G} = \mathbf{C}\mathbf{D}_G\mathbf{C}^{-1} \text{ where } \mathbf{D}_G = \begin{pmatrix} \lambda_1(\mathbf{G}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(\mathbf{G}) \end{pmatrix}$$

Since

$$\sum_{k\geq 0} \beta^k \mathbf{G}^k = \mathbf{C}\left(\sum_{k\geq 0} \beta^k \mathbf{D}_G^k\right)\mathbf{C}^{-1}$$

the sum converges if and only if $\sum_{k\geq 0}(\beta\lambda_{\max}(\mathbf{G}))^k$ converges, i.e. $\beta\lambda_{\max}(\mathbf{G}) < 1$.

Proof of Proposition

Simple way of proving this Proposition.

This game is a **potential game** (Monderer and Shapley, 1996)

A game is a potential game if there is a numerical function $P : X \to \mathbb{R}$ such that, for each $i \in N$, for each $x_{-i} \in X_{-i}$, and for each $x_i, z_i \in X_i$,

$$u_i(x_i, x_{-i}) - u_i(z_i, x_{-i}) = P(x_i, x_{-i}) - P(z_i, x_{-i})$$

Potential Games and Functions

Monderer and Shapley (GEB 1996) define *potential games* as games that admit a potential function P.

A function $P : X \rightarrow \mathbb{R}$ is called an *ordinal potential function* for the game if for each $i \in N$ and all $x_{-i} \in X_{-i}$, we have:

$$u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i}) \Longleftrightarrow P(x_i, x_{-i}) \geq P(x'_i, x_{-i})$$

$\forall x \in X$ and $\forall x_{-i} \in X_{-i}$

A *potential function* is a global function defined on the space of pure strategy profiles such that the change in any player's payoffs from a unilateral deviation is exactly matched by the change in the potential P.

A function $P : X \to \mathbb{R}$ is is called an *exact potential function* for the game if for each $i \in N$ and all $x_{-i} \in X_{-i}$, we have:

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = P(x_i, x_{-i}) - P(x_i', x_{-i})$$

$\forall x \in X$ and $\forall x_{-i} \in X_{-i}$

A finite game (or a game with a finite number of players but with infinite strategy spaces) is a **potential game** [ordinal potential game, exact potential game] if there exists a function $P : X \to \mathbb{R}$ such that $P(x_i, x_{-i})$ gives information about $u_i(x_i, x_{-i})$ for each i.

If so, P is referred to as the potential function.

The potential function has a natural analogy to "energy" in physical systems.

It will be useful both for locating pure strategy Nash equilibria and also for the analysis of "myopic" dynamic network formation models.

The potential function allows for a neat and explicit characterization of the stationary distribution of the Markov chain (as a Gibbs measure).

A finite game G is called an ordinal (exact) potential game if it admits an ordinal (exact) potential.

We refer to ordinal potential games as **potential games**, and only add the "exact" qualifier when this is necessary.

A game G with infinite strategy space and finite number of players is a potential game if it admits a continuous potential function.

When the payoff functions are twice continuously differentiable, Monderer and Shapley (1996) present a convenient characterization of potential games.

That is, a game is a potential game if and only if the cross partial derivatives of the utility functions for any two players are the same, i.e.,

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i})}{\partial x_i \partial x_j} = \frac{\partial^2 u_j(x_j, \mathbf{x}_{-j})}{\partial x_i \partial x_j} = \frac{\partial^2 P(x_i, \mathbf{x}_{-i})}{\partial x_i \partial x_j} \qquad \forall i, j$$

This equation can be used to identify potential games. If this holds, the potential function P can be calculated by integrating this equation.

Similar conditions hold for nondifferentiable payoff functions by replacing “differentials” with “differences”.

Theorem: Every potential game has at least one pure strategy Nash equilibrium.

Proof: The global maximum of an ordinal potential function is a pure strategy Nash equilibrium.

To see this, suppose that x^* corresponds to the global maximum.

Then, for any i, we have, by definition of a NE,

$$P(x_i^*, x_{-i}^*) \geq P(x_i', x_{-i}^*) \text{ for all } x_i \in X_i$$

But since P is a potential function,

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i', x_{-i}^*) \iff P(x_i^*, x_{-i}^*) \geq P(x_i', x_{-i}^*) \text{ for all } x_i \in X_i$$

Therefore, $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i', x_{-i}^*)$ for all $x_i \in X_i$ and for all i. Hence x_i^* is a pure strategy Nash equilibrium.

Note, however, that there may also be other pure strategy Nash equilibria corresponding to local maxima.

Example: Cournot

Consider a symmetric oligopoly Cournot competition with linear cost functions $c_i(q_i) = cq_i$, $1 \leq i \leq n$.

Linear inverse demand function: $F(Q) = a - bQ$

Payoff (Profit)

$$u_i(q_i, q_{-i}) = F(Q)q_i - cq_i = aq_i - b\sum_{j=1}^{j=n} q_iq_j - cq_i$$

Define the Potential function as:

$$\begin{aligned}
P(q_i, q_{-i}) &= \sum_{i=1}^{i=n} u_i(q_i, q_{-i}) - \frac{b}{2}\sum_{i=1}^{i=n}\sum_{j=1,j\neq i}^{j=n} q_i q_j \\
&= a\sum_{i=1}^{i=n} q_i - b\sum_{i=1}^{i=n} q_i^2 - b\sum_{i=1}^{i=n}\sum_{j=1,j\neq i}^{j=n} q_i q_j - c\sum_{i=1}^{i=n} q_i \\
&\quad -\frac{b}{2}\sum_{i=1}^{i=n}\sum_{j=1,j\neq i}^{j=n} q_i q_j \\
&= a\sum_{i=1}^{i=n} q_i - b\sum_{i=1}^{i=n} q_i^2 - c\sum_{i=1}^{i=n} q_i - \frac{b}{2}\sum_{i=1}^{i=n}\sum_{j=1,j\neq i}^{j=n} q_i q_j
\end{aligned}$$

Here the potential $P(q_i, q_{-i})$ is constructed by taking the sum of all utilities, a sum that is corrected by a term which takes into account the externalities exerted by each agent i.

Determine

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i})$$

$$\begin{aligned}
&= aq_i - b\sum_{j=1}^{j=n} q_i q_j - cq_i - \left(aq_i' - b\sum_{j=1}^{j=n} q_i' q_j - cq_i'\right) \\
&= (a-c)\left(q_i - q_i'\right) - bq_i\left(\sum_{j=1, j\neq i}^{j=n} q_j + q_i\right) + bq_i'\left(\sum_{j=1, j\neq i}^{j=n} q_j + q_i'\right) \\
&= (a-c)\left(q_i - q_i'\right) - b\left(q_i^2 - q_i'^2\right) - b\sum_{j=1, j\neq i}^{j=n}\left(q_i q_j - q_i' q_j\right)
\end{aligned}$$

Similarly

$$
\begin{aligned}
& P(q_i, q_{-i}) - P(q_i', q_{-i}) \\
= \;& (a-c)\left(\sum_{i=1}^{i=n} q_i - \sum_{i=1}^{i=n} q_i'\right) - b\left(\sum_{i=1}^{i=n} q_i^2 - \sum_{i=1}^{i=n} q_i'^2\right) \\
& -\frac{b}{2}\left(\sum_{i=1}^{i=n}\sum_{j=1,j\neq i}^{j=n} q_i q_j - \sum_{i=1}^{i=n}\sum_{j=1,j\neq i}^{j=n} q_i' q_j\right) \\
= \;& (a-c)\left(q_i - q_i'\right) - b\left(q_i^2 - q_i'^2\right) - \frac{b}{2}\sum_{j=1,j\neq i}^{j=n} q_j \left(\sum_{i=1}^{i=n} q_i - \sum_{i=1}^{i=n} q_i'\right) \\
= \;& (a-c)\left(q_i - q_i'\right) - b\left(q_i^2 - q_i'^2\right) - \frac{b}{2}\sum_{j=1,j\neq i}^{j=n} \left(q_i q_j - q_i' q_j\right)
\end{aligned}
$$

Thus $P : X \to \mathbb{R}$ is an *ordinal potential function* for the game if for each $i \in N$ and all $q_{-i} \in X_{-i}$ since we have:

$$u_i(q_i, q_{-i}) \geq u_i(q_i', q_{-i}) \Longleftrightarrow P(q_i, q_{-i}) \geq P(q_i', q_{-i})$$

since

$$\begin{aligned}
(a-c)\left(q_i - q_i'\right) - b\left(q_i^2 - q_i'^2\right) &\geq b \sum_{j=1, j\neq i}^{j=n} \left(q_i q_j - q_i' q_j\right) \\
&\geq \frac{b}{2} \sum_{j=1, j\neq i}^{j=n} \left(q_i q_j - q_i' q_j\right)
\end{aligned}$$

Exact Potential function:

$$P(q_i, q_{-i}) = a \sum_{i=1}^{i=n} q_i - b \sum_{i=1}^{i=n} q_i^2 - b \sum_{i=1}^{i=n} \sum_{j=1, j \neq i}^{j=n} q_i q_j - c \sum_{i=1}^{i=n} q_i$$

Indeed

$$
\begin{aligned}
& P(q_i, q_{-i}) - P(q_i', q_{-i}) \\
= \;& (a-c)\left(q_i - q_i'\right) - b\left(q_i^2 - q_i'^2\right) - b \sum_{j=1, j\neq i}^{j=n} q_j \left(\sum_{i=1}^{i=n} q_i - \sum_{i=1}^{i=n} q_i' \right) \\
= \;& (a-c)\left(q_i - q_i'\right) - b\left(q_i^2 - q_i'^2\right) - b \sum_{j=1, j\neq i}^{j=n} \left(q_i q_j - q_i' q_j\right) \\
= \;& u_i(q_i, q_{-i}) - u_i(q_i', q_{-i})
\end{aligned}
$$

In that case

$$\frac{\partial u_i(q_i, \mathbf{q}_{-i})}{\partial q_i} = a - c - 2bq_i - b \sum_{j=1, j \neq i}^{j=n} q_j \text{ and } \frac{\partial^2 u_i(q_i, \mathbf{q}_{-i})}{\partial q_i \partial q_j} = -b$$

$$\frac{\partial P(q_i, \mathbf{q}_{-i})}{\partial q_i} = a - c - 2bq_i - b \sum_{j=1, j \neq i}^{j=n} q_j \text{ and } \frac{\partial^2 P(q_i, \mathbf{q}_{-i})}{\partial q_i \partial q_j} = -b$$

In our case

$$u_i(x_i, x_{-i}) = \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j$$

For simplicity assume $\gamma = 0$ and $\beta = 1$ so that

$$u_i(x_i, x_{-i}) = \alpha x_i - \frac{1}{2} x_i^2 + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j$$

Define the Potential function as:

$$P(x_i, x_{-i}, g)$$

$$\begin{aligned}
&= \sum_{i=1}^{i=n} u_i(x_i, x_{-i}) - \frac{\lambda}{2} \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} g_{ij} x_i x_j \\
&= \alpha \sum_{i=1}^{i=n} x_i - \frac{1}{2} \sum_{i=1}^{i=n} x_i^2 + \lambda \sum_{i=1}^{i=n} \sum_{j=1}^{n} g_{ij} x_i x_j - \frac{\lambda}{2} \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} g_{ij} x_i x_j \\
&= \alpha \sum_{i=1}^{i=n} x_i - \frac{1}{2} \sum_{i=1}^{i=n} x_i^2 + \frac{\lambda}{2} \sum_{i=1}^{i=n} \sum_{j=1}^{n} g_{ij} x_i x_j
\end{aligned}$$

Matrix form

$$
\begin{aligned}
P(x_i, x_{-i}, g) &= \alpha \mathbf{x}^{\mathsf{T}} \mathbf{1} - \frac{1}{2} \mathbf{x}^{\mathsf{T}} \mathbf{x} + \mathbf{x}^{\mathsf{T}} \frac{\lambda}{2} \mathbf{G} \mathbf{x} \\
&= \alpha \mathbf{x}^{\mathsf{T}} \mathbf{1} - \frac{1}{2} \mathbf{x}^{\mathsf{T}} (\mathbf{I} - \lambda \mathbf{G}) \mathbf{x}
\end{aligned}
$$

Here the potential $P(x_i, x_{-i}, g)$ is constructed by taking the sum of all utilities, a sum that is corrected by a term which takes into account the *network externalities* exerted by each agent i.

It is well-known (see e.g., Monderer and Shapley, 1996) that the set of solutions of the program $\max_{\mathbf{x}} P(x_i, x_{-i}, g)$ forms a subset of the set of Nash equilibria of this game.

This program has a unique interior solution if the potential function $P(x_i, x_{-i}, g)$ is strictly concave on the relevant domain.

Nash Equilibrium: Potential Function

- Unique, stable, etc.?

- Reformulate equilibrium conditions as a max problem:

- A *potential function* φ for a game with payoffs V_i

$$\varphi(x_i, \mathbf{x}_{-i}) - \varphi(x_i', \mathbf{x}_{-i}) = V_i(x_i, \mathbf{x}_{-i}) - V_i(x_i', \mathbf{x}_{-i})$$

for all x_i, x_i' and for all i. [Monderer & Shapley (1996)]

- Game with quadratic payoffs, $\tilde{U}_i$, has an exact potential:

$$\varphi(\mathbf{x}) = \sum_i[(x_i - \tfrac{1}{2}x_i^2) - \tfrac{1}{2}\delta\sum_{i,j}g_{ij}x_ix_j]$$

Nash Equilibrium: Potential Function

- **Proposition**

  $\mathbf{x}$ *is a Nash equilibrium of any game with best response* $f_i(\mathbf{x})$ *iff* $\mathbf{x}$ *satisfies the Kuhn-Tucker conditions of the problem*

$$\max \varphi(\mathbf{x}) \quad s.t \quad x_i \geq 0 \quad \forall\, i$$

- **Proof**
  - $\mathbf{x}$ is equilibrium for game with payoffs $\tilde{U}_i$
    - no player has an incentive to deviate, since at $\mathbf{x}$, K-T conditions imply first-order conditions satisfied, and second order conditions satisfied $\varphi_{ii} < 0$
  - equilibria are same for games with best response $f_i(\mathbf{x})$

- Thus, the equilibria correspond to the maxima of the potential function.

The Hessian matrix of $P(x_i, x_{-i}, g)$ is easily computed to be $-(\mathbf{I}-\lambda\mathbf{G})$.

The matrix $\mathbf{I}-\lambda\mathbf{G}$ is positive definite if for all non-zero $\mathbf{x}$ we have

$$\mathbf{x}^\top (\mathbf{I}-\lambda\mathbf{G})\,\mathbf{x} > 0 \Leftrightarrow \lambda < \left(\frac{\mathbf{x}^\top\mathbf{G}\mathbf{x}}{\mathbf{x}^\top\mathbf{x}}\right)^{-1}$$

By the Rayleigh-Ritz theorem, we have $\mu_1(\mathbf{G}) = \sup_{\mathbf{x}\neq\mathbf{0}}\left(\frac{\mathbf{x}^\top\mathbf{G}\mathbf{x}}{\mathbf{x}^\top\mathbf{x}}\right)$.

Thus a necessary and sufficient condition for having a strict concave potential is that $\lambda\mu_1(\mathbf{G}) < 1$.

Example 2. Consider the network g in Figure 1.

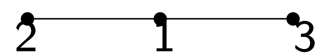


Figure 1

Largest eigenvalue: $2^{1/2}$. When $d\left(2^{1/2}\right) < c$, the unique Nash equilibrium is:

$$x_1^* = a\frac{c+2d}{c^2-2d^2} \qquad \text{and} \qquad x_2^* = x_3^* = a\frac{c+d}{c^2-2d^2}.$$

Network structure and peer effects

Standard *peer effects*, or intragroup externalities, are homogeneous across group members, an *average* influence.

Here, instead, the intragroup externality varies across group members with their network location:

$$x_i^* = \frac{b_i(\mathbf{g}, \lambda/\beta)}{b(\mathbf{g}, \lambda/\beta)} x^*.$$

The Bonacich network centrality captures the *variance* in peer effects.

Dyads

No social interactions.

Then, the utility of each agent i would be given by:

$$u_i(x_i) = \alpha x_i - \frac{1}{2}x_i^2$$

The unique symmetric equilibrium is:

$$x_{ni}^* = \alpha$$

Now, in order to understand the general model and to see the role of λ and γ, let us take the simplest possible network, that is $n = 2$ and each player has a link with the other, that is $g_{12} = g_{21} = 1$.

The adjacency matrix

$$\mathbf{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Two eigenvalues: $1, -1$. Thus $\mu_1(\mathbf{G}) = 1$.

The network locations in $\mathbf{g}$ are interchangeable. In this case, the utility is now given by:

$$u_i(x_1, x_2) = \alpha x_i - \frac{1}{2} x_i^2 - \gamma \left(x_i^2 + x_i x_j \right) + \lambda x_i x_j$$

where $0 \leq \gamma < 1$. Compared to our utility function $\beta = 1 + \gamma$.

The first order condition are:

$$\frac{\partial u_i}{\partial x_i} = \alpha - (1 + 2\gamma)\, x_i - (\gamma - \lambda)\, x_j = 0$$

Since we have a dyad, the unique symmetric equilibrium is given by:

$$x^* = \frac{\alpha}{1 - \lambda + 3\gamma}$$

Observe that since $\beta = 1 + \gamma$ and $\mu_1(\mathbf{G}) = 1$,

$$\beta > \lambda \mu_1(\mathbf{G}) \Leftrightarrow \lambda < 1 + \gamma$$

Guarantees this solution to be strictly positive.

Check with Theorem

$$\mathbf{x}^* = \frac{\alpha}{\beta + \gamma b(\mathbf{g}, \lambda/\beta)} \mathbf{b}(\mathbf{g}, \lambda/\beta)$$

Here

$$\begin{aligned}
\mathbf{b}(\mathbf{g}, a) &= \left[\mathbf{I} - \frac{\lambda}{\beta} \mathbf{G} \right]^{-1} \cdot \mathbf{1} \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\lambda}{\beta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -\lambda/\beta \\ -\lambda/\beta & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\beta^2}{\beta^2 - \lambda^2} & \frac{\lambda\beta}{\beta^2 - \lambda^2} \\ \frac{\lambda\beta}{\beta^2 - \lambda^2} & \frac{\beta^2}{\beta^2 - \lambda^2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\beta - \lambda} \\ \frac{\beta}{\beta - \lambda} \end{pmatrix}
\end{aligned}$$

Thus

$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \frac{\alpha}{\beta + \gamma b(\mathbf{g}, \lambda/\beta)} \begin{pmatrix} \frac{\beta}{\beta-\lambda} \\ \frac{\beta}{\beta-\lambda} \end{pmatrix}$$

where

$$b(\mathbf{g}, \lambda/\beta) = b_1(\mathbf{g}, \lambda/\beta) + b_2(\mathbf{g}, \lambda/\beta) = \frac{2\beta}{\beta - \lambda}$$

We have

$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\beta-\lambda+2\gamma} \\ \frac{\alpha}{\beta-\lambda+2\gamma} \end{pmatrix}$$

Now since $\beta = 1 + \gamma$, we have:

$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{1-\lambda+3\gamma} \\ \frac{\alpha}{1-\lambda+3\gamma} \end{pmatrix}$$

Suppose first that $\gamma = 0$, i.e. there is no global substituability. We obtain

$$x^* = \frac{\alpha}{1 - \lambda}$$

In the dyad, agents rip complementarities from their partner, and choose an effort level above the optimal value for an isolated agent ($x^* = \alpha$). The factor $1/(1 - \lambda) > 1$ is often referred to as the **social multiplier**.

Suppose now that $\lambda = 0$. We obtain

$$x^* = \frac{\alpha}{1 + 3\gamma}$$

Equilibrium efforts are decreasing in γ. Indeed, global substituabilities add to the idiosyncratic concavity in one's efforts, an exhaust decreasing marginal returns below the optimal value for an isolated agent. The general expression results from a combination of both effects.

Ex ante Heterogeneity in Games on Networks

More general network game with linear quadratic payoffs.

$N = \{1, \ldots, n\}$ is a finite set of agents.

Each agent $i \in N$ selects $z_i \geq 0$. Payoffs are:

$$u_i(\mathbf{z}) = \alpha_i z_i + \frac{1}{2}\sigma_{ii} z_i^2 + \sum_{j \neq i} \sigma_{ij} z_i z_j.$$

Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ and $\boldsymbol{\Sigma} = [\sigma_{ij}]$.

Game $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\Sigma})$ with players in N such that $\boldsymbol{\alpha} > \mathbf{0}$ (that is, $\alpha_i > 0$, for all $i \in N$) and $\sigma_{ii} < \min\{0, \min\{\sigma_{ij} : j \neq i\}\}$, for all $i \in N$.

We further assume that $\sigma_{ii} = \sigma_{11}$, for all $i \in N$. This is without loss of generality.

Indeed, let $\mathbf{D} = diag(1, \sigma_{11}/\sigma_{22}, ..., \sigma_{11}/\sigma_{nn})$. This is a diagonal matrix with a strictly positive diagonal. It is readily checked that the Nash equilibria of $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\Sigma})$ and that of $\Gamma(\mathbf{D}\boldsymbol{\alpha}, \mathbf{D}\boldsymbol{\Sigma})$ coincide, where the diagonal terms of $\mathbf{D}\boldsymbol{\Sigma}$ are all equal to σ_{11}.

Let $\mathbf{I}$ be the identity matrix and $\mathbf{J}$ the matrix of ones.

Additive decomposition of the interaction matrix:

$$\mathbf{\Sigma} = -\beta \mathbf{I} - \gamma \mathbf{J} + \lambda \mathbf{G}.$$

Own-concavity effects $-\beta \mathbf{I}$

Global substitutability effects $-\gamma \mathbf{J}$

Local (network) complementarity effects $+\lambda \mathbf{G}$.

Following this decomposition, payoffs can now be rewritten as:

$$u_i(\mathbf{z}) = \alpha_i z_i - \frac{1}{2} (\beta - \gamma) z_i^2 - \gamma \sum_{j=1}^{n} z_i z_j + \lambda \sum_{j=1}^{n} g_{ij} z_i z_j$$

Definition 0.1 *Given a vector* $\mathbf{u} \in \mathbb{R}^n_+$, *and* $a \geq 0$ *a small enough scalar, we define the vector of* $\mathbf{u}$*-weighted centrality of parameter* a *in the network* $\mathbf{g}$ *as:*

$$\mathbf{w_u}\left(\mathbf{g}, a\right) = \left(\mathbf{I} - a\mathbf{G}^{-1}\right)\mathbf{u} = \sum_{p=0}^{+\infty} a^p \mathbf{G}^p \mathbf{u}.$$

Katz-Bonacich centrality $\mathbf{b}\left(\mathbf{g}, a\right)$ corresponds to the $\mathbf{u}$-weighted centrality with $\mathbf{u} = \mathbf{1}$ (where $\mathbf{1}$ is the vector of ones)

Denote by $\omega(\mathbf{G})$ the largest eigenvalue of $\mathbf{G}$. For all vector $\mathbf{u} \in \mathbb{R}^n$, let $u = u_1 + ... + u_n$.

Theorem 0.1 *Consider a game* $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\Sigma})$ *with* $\boldsymbol{\alpha} > \mathbf{0}$ *and* $\boldsymbol{\Sigma}$ *is decomposed additively.*

(a) *Suppose first that* $\boldsymbol{\alpha} = \alpha\mathbf{1}$. *Then,* $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\Sigma})$ *has a unique Nash equilibrium in pure strategies if and only if* $\beta > \lambda\omega(\mathbf{G})$. *This equilibrium* $\mathbf{z}^*$ *is interior and given by:*

$$\mathbf{z}^* = \frac{\alpha}{\beta + \gamma w_{\mathbf{1}}(\mathbf{g}, \lambda/\beta)} \mathbf{w}_{\mathbf{1}}(\mathbf{g}, \lambda/\beta). \tag{1}$$

(b) *Suppose now that* $\boldsymbol{\alpha} \neq \alpha \mathbf{1}$. *Let* $\alpha^{\mathsf{max}} = \mathsf{max}\ \{\alpha_i \mid i \in N\}$ *and* $\alpha_{\mathsf{min}} = \mathsf{min}\{\alpha_i \mid i \in N\}$, *with* $\alpha^{\mathsf{max}} > \alpha_{\mathsf{min}} > 0$. *If* $\beta > \lambda\omega(\mathbf{G}) + n\gamma(\alpha^{\mathsf{max}}/\alpha_{\mathsf{min}} - 1)$, *then* $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\Sigma})$ *has a unique Nash equilibrium in pure strategies* $\mathbf{z}^*$, *which is interior and given by:*

$$\mathbf{z}^* = \frac{1}{\beta}\left[\mathbf{w}_{\boldsymbol{\alpha}}(\mathbf{g}, \lambda/\beta) - \frac{\gamma w_{\boldsymbol{\alpha}}(\mathbf{g}, \lambda/\beta)}{\beta + \gamma w_{\mathbf{1}}(\mathbf{g}, \lambda/\beta)}\mathbf{w}_{\mathbf{1}}(\mathbf{g}, \lambda/\beta)\right]. \tag{2}$$

When $\boldsymbol{\alpha} = \alpha\mathbf{1}$, the equilibrium existence, uniqueness (and interiority) condition is independent of γ, the global level of substitutabilities, and only depends on the own concavity term β and the network of local complementarities $\lambda\mathbf{G}$.

For general $\boldsymbol{\alpha}$, a necessary and sufficient condition for equilibrium existence and uniqueness is that $-\boldsymbol{\Sigma}$ has all its principal minors strictly positive, that is, $-\boldsymbol{\Sigma}$ is a $P-$matrix in the language of the linear complementarity problem.

The $P-$matrix condition does not guarantee that the equilibrium is interior. Besides, the $P-$matrix property is computationally very demanding and economically nonintuitive.

Comparative statics

Theorem. Let $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ symmetric s.t. $(\alpha, \beta, \gamma, \lambda) = (\alpha', \beta', \gamma', \lambda')$ and $\mathbf{G} \leq \mathbf{G}'$. If $\beta > \lambda\mu_1(\mathbf{G}')$, then $x^*(\mathbf{\Sigma}') > x^*(\mathbf{\Sigma})$.

The denser the pattern of local complementarities, the higher the aggregate activity level.

In words, the denser the pattern of local complementarities, the higher the aggregate outcome, as players can rip more complementarities in $\mathbf{g}'$ than in $\mathbf{g}$.

The geometric intuition for this result is clear. Recall that $b(\mathbf{g}, \lambda^*)$ counts the total number of weighted paths in $\mathbf{g}$. This is obviously an increasing function in $\mathbf{g}$ (for the inclusion ordering), as more links imply more such paths.

Application 1 : Crime networks

There are n criminals, each exerting a level of crime x_i that results from a trade off between the costs and benefits of criminal activities.

The expected utility of criminal i is:

$$u_i(\mathbf{x}, \mathbf{g}) = y_i(\mathbf{x}) - p_i(\mathbf{x}, \mathbf{g})f, \qquad (1)$$

$y_i(\mathbf{x})$ are the proceeds, $p_i(\mathbf{x}, \mathbf{g})$ the apprehension probability, and f the corresponding fine.

The cost of committing crime $p_i(\mathbf{x}, \mathbf{g})f$ increases with x_i, as the apprehension probability increases with one's involvement in crime, hitherto, with one's exposure to deterrence.

Also, and consistent with standard criminology theories, criminals improve illegal practice through interactions with their direct criminal mates.

Formally, criminals are connected through a friendship network $\mathbf{r}$, where $r_{ij} = 1$ when i and j are directly related to each other. For instance, let:

$$\begin{cases} y_i\left(\mathbf{x}\right) = x_i\left[\mathbf{1} - \eta \sum_{j=1}^{n} x_j\right] \\ p_i(\mathbf{x}, \mathbf{g}) = p_0 x_i\left[\mathbf{1} - \nu \sum_{j=1}^{n} g_{ij} x_j\right] \end{cases} .$$

The expected utility then becomes:

$$u_i(\mathbf{x}, \mathbf{g}) = (1 - \pi)x_i - \eta \sum_{j=1}^{n} x_i x_j + \pi\nu \sum_{j=1}^{n} g_{ij} x_i x_j, \tag{2}$$

where $\pi = p_0 f$ is the marginal expected punishment cost for an isolated criminal, and $-\eta < 0$ captures a congestion in the crime market.

The utility function (2) coincides with the expression our general utility with $\alpha = 1 - \pi$, $\beta = \gamma = \eta$ and $\lambda = \pi\nu$.

When $\pi\nu\mu_1(\mathbf{g}) < \eta$, the unique Nash equilibrium of the crime game with payoffs (2) is:

$$\mathbf{x}^* = \frac{1 - \pi}{\eta} \frac{1}{1 + b(\mathbf{g}, \pi\frac{\nu}{\eta})} \mathbf{b}(\mathbf{g}, \pi\frac{\nu}{\eta}).$$

Application 2 : R&D collaboration networks

R&D partnerships have become a widespread phenomenon

Consider a standard Cournot game with n (ex ante) identical firms, each of them choosing the quantity q_i.

As in Goyal and Moraga-González (2001) and Goyal and Joshi (2003), firms can form bilateral agreements to jointly invest in cost-reducing R&D activities.

We set $g_{ij} = 1$ when firms i and j set up a collaboration link.

Firm i's marginal cost is:

$$c_i(\mathbf{q}, \mathbf{g}) = \lambda_0 - \lambda \sum_{j \neq i} g_{ij} q_j$$

Here, $\lambda_0 > 0$, represents the marginal cost of an isolated firm, while $\lambda > 0$ is the cost reduction induced by each link it forms.

p_i is the price of good i. This gives the inverse demand function for firm i

$$p_i = \phi - \sum_{j=1}^{n} q_j$$

The profit function of firm i is:

$$\begin{aligned}
\pi_i(\mathbf{q}, \mathbf{g}) &= p_i q_i - c_i(\mathbf{q}, \mathbf{g}) q_i \\
&= \left[\phi - \sum_{j=1}^{n} q_j\right] q_i - \left[\lambda_0 - \lambda \sum_{j \neq i} g_{ij} q_j\right] q_i \\
&= (\phi - \lambda_0)\, q_i - \sum_{j=1}^{n} q_i q_j + \lambda \sum_{j \neq i} g_{ij} q_i q_j.
\end{aligned}$$

Again, this objective function is a particular case of our general utility, where $\alpha = \phi - \lambda_0 > 0$ and $\beta = \gamma = 1$.

We conclude that the Cournot game has a unique Nash equilibrium in pure strategies:

$$\mathbf{q}^* = \frac{(\phi - \lambda_0)}{1 + b(\mathbf{g}, \lambda)} \mathbf{b}(\mathbf{g}, \lambda),$$

when $1 > \lambda \mu_1(\mathbf{g})$.

In particular, the comparative statics Theorem implies that the overall industry output increases when the network of collaboration links expands, irrespective of this network geometry and the number of additional links.

For the case of a linear inverse demand curve, this generalizes the findings in Goyal and Moraga-González (2001) and Goyal and Joshi (2003), where monotonicity of industry output is established for the case of regular collaboration networks, where each firm forms the same number of bilateral agreements.

For such regular networks, links are added as multiples of n, as all firms' connections are increased simultaneously.

Application 3 : International trade networks

Consider a set of countries $\mathcal{N} = \{1, 2, \ldots, n\}$ and a network $\mathbf{g}$ representing links between them.

A link indicates the presence of an (import or export) *trade* relationship between two countries.

Each country i provides a volume $x_i \geq 0$ of trade.

Countries are local monopolists and the inverse demand function for country $i \in \mathcal{N}$ is given by:

$$p_i = 1 - \theta x_i$$

with a parameter $\theta > 0$.

Products produced by different countries are not substitutable.

The marginal cost of production of each country i is $c_i(\mathbf{x}, \mathbf{g}) \geq 0$.

$$c_i(\mathbf{x}, \mathbf{g}) = c_0 - \lambda \sum_{j=1}^{n} a_{ij} x_j,$$

where $c_0 > 0$ represents a country's marginal cost when it has no links, and $\lambda > 0$ is the cost reduction induced by each trade relationship formed by a country.

Production costs decrease with the volume of trade of the trading partner due to technology spillovers

The profit function for country i is π_i in a trade network $\mathbf{g}$ is given by

$$\begin{aligned}
\pi_i(\mathbf{q}, \mathbf{g}) &= p_i q_i - c_i(\mathbf{q}, \mathbf{g}) q_i \\
&= \left[\phi - \sum_{j=1}^{n} q_j\right] q_i - \left[\lambda_0 - \lambda \sum_{j \neq i} g_{ij} q_j\right] q_i \\
&= (\phi - \lambda_0)\, q_i - \sum_{j=1}^{n} q_i q_j + \lambda \sum_{j \neq i} g_{ij} q_i q_j.
\end{aligned}$$

Unique Nash equilibrium: Bonacich centrality

Application 4 : Conformism and social norms

Each individual has a utility that depends on the difference between her behavior and that of her reference group.

Each individual chooses an action $x_i \geq 0$ and loses utility when failing to conform to the social norm of her reference group, which is equal to the average action of its members.

The network $N = \{1, \ldots, n\}$ is a finite set of individuals. Individuals are connected by a network of social connections.

We represent social connections by a graph/network $\mathbf{g}$. To any network $\mathbf{g}$, we can associate its adjacency matrix, that we denote by $\mathbf{G}$.

The coefficients of the matrix $\mathbf{G}$ are the g_{ij}s, $1 \leq i, j \leq n$. When i and j are friends we set $g_{ij} = 1$. Let also $g_{ii} = 0$ for all i. Thus, by definition, each cell in $\mathbf{G}$ takes on values zero or one. Given our convention that $g_{ii} = 0$, the diagonal of $\mathbf{G}$ consists on zeros. Since $g_{ij} = g_{ji}$, the matrix $\mathbf{G}$ is symmetric.

Each player i has $g_i = \sum_{j=1}^{n} g_{ij}$ direct links in $\mathbf{g}$, and thus the average action of her friends, that is the action of her reference group, is given by:

$$x_i^{av} = \frac{1}{g_i} \sum_{j=1, j \neq i}^{n} g_{ij} x_j = \sum_{j=1, j \neq i}^{n} g_{ij}^{*} x_j$$

Preferences Each individual $i = 1, ..., n$ selects an effort/action $x_i \geq 0$, and obtains a payoff $u_i(\mathbf{x}, \mathbf{g})$, given by the following utility function, with $\xi, \alpha, \theta, d > 0$:

$$u_i(\mathbf{x}, \mathbf{g}) = \xi + \alpha x_i - \theta x_i^2 - d(x_i - x_i^{av})^2 \qquad (3)$$

The utility function (3) is such that each individual wants to minimize the social distance between herself and her reference group, where d is the parameter describing the taste for conformity.

Indeed, the individual loses utility $d(x_i - x_i^{av})^2$ from failing to conform to others.

The average action of the reference group of agent i, x_i^{av}, explicitly depends on the underlying network structure, and thus each agent has a different x_i^{av} depending on her location in the network.

Bilateral influences of this utility function.

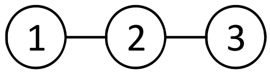
$$\frac{\partial^2 u_i(\mathbf{x},\mathbf{g})}{\partial x_i \partial x_j} = \begin{cases} -2(\theta + d), \text{ when } i = j \\ 0, \text{ when } i \neq j \text{ and } g_{ij} = 0 \\ 2d/g_i > 0, \text{ when } i \neq j \text{ and } g_{ij} = 1 \end{cases} .$$

Since, when $i \neq j$, $\sigma_{ij} > 0$, an increase in effort from j triggers a downwards shift in i's response and thus efforts are strategic complements from i's perspective within the pair (i, j).

This utility function (3) thus coincides with

$$u_i = \alpha x_i - \frac{1}{2}(\beta - \gamma)\, x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij}^* x_i x_j$$

with $\beta = 2(\theta + d)$, $\gamma = 0$, $\lambda = 2d$ and $g_{ij}^* = g_{ij}/g_i$. Note that $\mathbf{g}^*$ is a row-normalization of the initial friendship network $\mathbf{g}$, as illustrated in the following example, where $\mathbf{G}$ and $\mathbf{G}^*$ are the adjacency matrices of, respectively, $\mathbf{g}$ and $\mathbf{g}^*$.



$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{G}^* = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

Observe that $\mathbf{G}^*$ is a stochastic matrix, that is $g^*_{ij} \geq 0$ and $\sum_j g^*_{ij} = 1$. This implies that $\mu_1(\mathbf{G}^*) = 1$ and $\mathbf{G}^{*k}$ is also a stochastic matrix, that is $g^{*[k]}_{ij} \geq 0$ and $\sum_j g^{*[k]}_{ij} = 1$, $\forall k$.

Applying our Theorem, it is easy to see that this conformity game with payoffs (3) has a unique Nash equilibrium in pure strategies and, whatever the structure of the network, this equilibrium is always symmetric, that is $x^* = x^*_1 = ... = x^*_n$ and $x^{av*}_i = x^{av*}_{i1} = ... = x^{av*}_{in}$, and is given by:

$$x^* = x^{av*}_i = \frac{\alpha}{2\theta} \tag{4}$$

In particular, the equilibrium Bonacich-centrality measure is the same for all individuals and is equal to:

$$b_1(\mathbf{g}^*, \frac{d}{\theta + d}) = ... = b_n(\mathbf{g}^*, \frac{d}{\theta + d}) = \frac{\theta + d}{\theta}$$

To prove this result, one has to calculate the Bonacich vector since it is the only source of heterogeneity between players. In a conformist game, we have:

$$\begin{aligned}
b_i(\mathbf{g}^*, a) &= m_{ii}(\mathbf{g}^*, a) + \sum_{j \neq i} m_{ij}(\mathbf{g}^*, a) \\
&= a \sum_{j=1}^{n} g_{ij}^* + ... + a^k \sum_{j=1}^{n} g_{ij}^{*[k]} + ... \\
&= \sum_{j=1}^{+\infty} a^k = \frac{1}{1-a}
\end{aligned}$$

The equilibrium value (4) is exactly the value found by Akerlof (1997), page 1010.

Even if individuals are ex ante heterogeneous because of their location in the network, in a conformist equilibrium where each individual would like to conform as much as possible to the norm of her reference group, all individuals will exert the same effort level.

The distribution of population does not matter in equilibrium even if it matters ex ante. The only relevant statistics is the average.

Local aggregate or local average?

1. The local-aggregate network model

An individual's utility depends on the *aggregate* effort level of her friends.

$y_{i,r}$ the effort level of individual i in network r

$Y_r = (y_{1,r}, ..., y_{n_r,r})'$ the population effort profile in network r.

Utility:

$$u_{i,r}(y_{i,r}) = \left(a_{i,r} + \eta_r + \epsilon_{i,r}\right) y_{i,r} - \frac{1}{2} y_{i,r}^2 + \phi_1 \sum_{j=1}^{n_r} g_{ij,r} y_{i,r} y_{j,r} \tag{5}$$

where $\phi_1 \geq 0$.

Best-reply function for individual i,

$$y_{i,r} = \phi_1 \sum_{j=1}^{n_r} g_{ij,r} y_{j,r} + a_{i,r} + \eta_r + \epsilon_{i,r}. \qquad (6)$$

Denote: $\pi_{i,r} = a_{i,r} + \eta_r + \epsilon_{i,r}$, $\Pi_r = (\pi_{1,r}, \cdots, \pi_{n_r,r})'$, and $g_r^{\max} = \max_i g_{i,r}$ the highest degree of network r.

Proposition 1 *If* $0 \leq \phi_1 g_r^{\max} < 1$, *then the network game with payoffs (5) has a unique interior Nash equilibrium in pure strategies given by (6). In matrix form, this can be written as:*

$$Y_r = (I_{n_r} - \phi_1 G_r)^{-1} \Pi_r. \qquad (7)$$

2. The local-average network model

The *average effort level* of friends affects an individual's utility.

Denote

$$y_{i,r} = \sum_{j \in N_{i,r}} g^{*}_{ij,r} y_{j,r}$$

the average effort of individual i's friends.

Payoff

$$u_{i,r}(y_{i,r}) = \left(a^{*}_{i,r} + \eta^{*}_{r} + \epsilon^{*}_{i,r}\right) y_{i,r} - \frac{1}{2} y^{2}_{i,r} - \frac{d}{2} (y_{i,r} - \overline{y}_{i,r})^{2} \tag{8}$$

with $d \geq 0$.

Best-reply function of i:

$$y_{i,r} = \phi_2 \sum_{j=1}^{n_r} g^*_{ij,r} y_{j,r} + a_{i,r} + \eta_r + \epsilon_{i,r}, \tag{9}$$

where $\phi_2 = d/(1+d)$, $a_{i,r} = (1-\phi_2)a^*_{i,r}$, $\eta_r = (1-\phi_2)\eta^*_r$, and $\epsilon_{i,r} = (1-\phi_2)\epsilon^*_{i,r}$.

Proposition 2 *If $0 \leq \phi_2 < 1$, then the network game with payoffs (8) has a unique interior Nash equilibrium in pure strategies given by (9). In matrix form, this can be written as:*

$$Y_r = (I_{n_r} - \phi_2 G^*_r)^{-1} \Pi_r. \tag{10}$$

3. The hybrid network model

Integrating *local-aggregate* and *local-average* effects into the same model.

Utility function

$$u_{i,r}(y_{i,r}) = \left(a^*_{i,r} + \eta^*_r + \epsilon^*_{i,r}\right) y_{i,r} - \frac{1}{2} y^2_{i,r}$$

$$+ d_1 \sum_{j=1}^{n_r} g_{ij,r} y_{i,r} y_{j,r} - \frac{d_2}{2} (y_{i,r} - \overline{y}_{i,r})^2, \qquad (11)$$

where $d_1 \geq 0$ and $d_2 \geq 0$.

Best-reply function of individual i is:

$$y_{i,r} = \phi_1 \sum_{j=1}^{n_r} g_{ij,r} y_{j,r} + \phi_2 \sum_{j=1}^{n_r} g^*_{ij,r} y_{j,r} + a_{i,r} + \eta_r + \epsilon_{i,r} \tag{12}$$

where $\phi_1 = d_1/(1 + d_2)$, $\phi_2 = d_2/(1 + d_2)$, $a_{i,r} = (1 - \phi_2)a^*_{i,r}$, $\eta_r = (1 - \phi_2)\eta^*_r$, and $\epsilon_{i,r} = (1 - \phi_2)\epsilon^*_{i,r}$.

Proposition 3 *If $\phi_1 \geq 0$, $\phi_2 \geq 0$ and $\phi_1 g_r^{\mathsf{max}} + \phi_2 < 1$, then the network game with payoffs (11) has a unique interior Nash equilibrium in pure strategies given by (12). In matrix form, this can be written as:*

$$Y_r = (I_{n_r} - \phi_1 G_r - \phi_2 G^*_r)^{-1} \Pi_r \tag{13}$$

Local aggregate versus local average

Local aggregate: Best-reply function for individual i,

$$y_{i,r} = \phi_1 \sum_{j=1}^{n_r} g_{ij,r} y_{j,r} + a_{i,r} + \eta_r + \epsilon_{i,r}.$$

or

$$Y_r = \phi_1 G_r Y_r + \Pi_r$$

Local average: Best-reply function for individual i,

$$y_{i,r} = \phi_2 \sum_{j=1}^{n_r} g^*_{ij,r} y_{j,r} + a_{i,r} + \eta_r + \epsilon_{i,r},$$

or

$$Y_r = \phi_1 G^*_r Y_r + \Pi_r$$

Network-based policies

The planner can fine-tune the exogenous payoff parameters. This only shifts the distribution of individual outcomes.

The planner can also manipulate the network geometry: the distribution of individual outcomes is now completely modified.

We focus on a particular policy, removing one node, and we identify the *optimal target* in the network, the *key player*.

The planner's problem: finding the Achille's heel

When i is removed, all cross effects with i vanish. We get $\mathbf{\Sigma^{-i}}$.

The impact on aggregate equilibrium outcome is twofold: less players (*direct* effect), a different matrix (*indirect*).

The planner's problem is:

$$\max_i \{x^*(\mathbf{\Sigma}) - x^*(\mathbf{\Sigma}^{-i})\} \quad \Leftrightarrow \quad \min_i \{b(\mathbf{g}^{-i}, \lambda/\beta)\}.$$

The network inter-centrality measure

The equilibrium Bonacich centrality fails to internalize all the network externalities, but the planner needs to internalize them all!

The inter-centrality keeps track of all the cross-contributions in Bonacich centralities across agents:

$$c_i(\mathbf{g}, a) = \frac{b_i^2(\mathbf{g}, a)}{m_{ii}(\mathbf{g}, a)}.$$

The Bonacich centrality of player i counts the number of paths in $\mathbf{g}$ that stem from i.

The intercentrality counts the total number of such paths that hit i;

It is the sum of i's Bonacich centrality and i's contribution to every other player's Bonacich centrality.

Holding $b_i(\mathbf{g},a)$ fixed, $c_i(\mathbf{g},a)$ decreases with the proportion of i's Bonacich centrality due to self-loops $m_{ii}(\mathbf{g},a)/b_i(\mathbf{g},a)$.

The key player

Theorem. If $\beta > \lambda\mu_1(\mathbf{G})$, the key player i^* is such that $c_{i^*}(\mathbf{g}, \lambda/\beta) \geq c_i(\mathbf{g}, \lambda/\beta)$, for all i.

The key player is the one with highest inter-centrality.

In fact, $c_i(\mathbf{g}, a) = b_i(\mathbf{g}, a) + \sum_{j \neq i}[b_j(\mathbf{g}, a) - b_j(\mathbf{g}^{-i}, a)]$.

Key players with ex ante heterogeneity α_i. For example, $\alpha_i = a_i + \eta + \epsilon_i$

Definition 1 *Assume that each agent has an ex ante heterogeneity of* α_i *for all* i. *Then, for all networks* g *and for all* i, *the intercentrality measure of a player* i *is:*

$$c_i(g, \phi, \boldsymbol{\alpha}) = \frac{b_{\boldsymbol{\alpha},i}(g, \phi) \sum_{j=1}^{n} m_{ji}(g, \phi)}{m_{ii}(g, \phi)} \tag{14}$$

where

$$\mathbf{b}_{\boldsymbol{\alpha}}(\mathbf{g}, \phi) = \sum_{k=0}^{+\infty} \phi^k \mathbf{G}^k \boldsymbol{\alpha} = [\mathbf{I} - \phi \mathbf{G}]^{-1} \boldsymbol{\alpha}$$

and the sum of weighted Bonacich centralities

$$b_{\boldsymbol{\alpha}}(\mathbf{g}, \phi) = \sum_{i=1}^{n} b_{\boldsymbol{\alpha},i}(\mathbf{g}, \phi) = \mathbf{1}^{\top} \mathbf{M} \boldsymbol{\alpha}$$

and

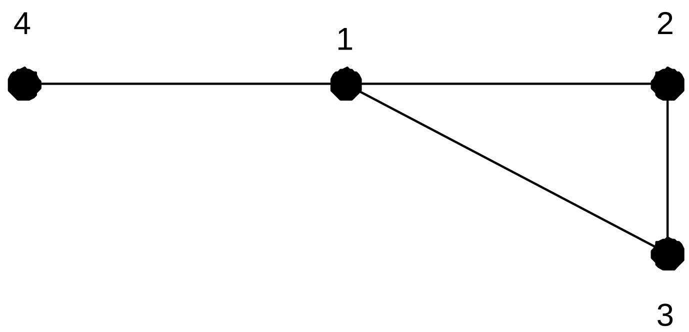
$$\mathbf{M}(\mathbf{g}, \phi) = [\mathbf{I} - \phi \mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} \phi^k \mathbf{G}^k.$$

Example

Network of four delinquents (i.e. $n = 4$) with

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.1, 0.2, 0.3, 0.4)$$

and



$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Decay factor $\phi = 0.3$.

Nash equilibrium:

$$\begin{pmatrix} y_1^* \\ y_2^* \\ y_3^* \\ y_4^* \end{pmatrix} = \begin{pmatrix} b_{\alpha,1}(g,\phi) \\ b_{\alpha,2}(g,\phi) \\ b_{\alpha,3}(g,\phi) \\ b_{\alpha,4}(g,\phi) \end{pmatrix} = \begin{pmatrix} 0.66521 \\ 0.60377 \\ 0.68068 \\ 0.59958 \end{pmatrix}$$

Total crime effort:

$$y^* = y_1^* + y_2^* + y_3^* = b_\alpha(g,\phi) = 2.549$$

Delinquent 3 has the highest weighted Bonacich and thus provides the highest crime effort.

Intercentrality: $d_{i^*}(g, \phi) = b_\alpha(g, \phi) - b_\alpha^{[-i]}(g, \phi)$

Remove delinquent 1.

We have now a network with three delinquents, with $(\alpha_2, \alpha_3, \alpha_4) = (0.2, 0.3, 0.4)$ and where

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using the same decay factor, $\phi = 0.3$, we obtain:

$$\begin{pmatrix} y_2^* \\ y_3^* \\ y_4^* \end{pmatrix} = \begin{pmatrix} b_{\alpha,2}(g^{[-1]}, \phi) \\ b_{\alpha,3}(g^{[-1]}, \phi) \\ b_{\alpha,4}(g^{[-1]}, \phi) \end{pmatrix} = \begin{pmatrix} 0.31868 \\ 0.3956 \\ 0.4 \end{pmatrix}$$

so that the total effort is now given by:

$$y^{*[-1]} = y_2^* + y_3^* + y_4^* = b_\alpha^{[-1]}(g, \phi) = 1.114$$

Thus, player 1's contribution is

$$b_{\alpha}(g,\phi) - b_{\alpha}^{[-1]}(g,\phi) = 2.549 - 1.114 = 1.435$$

Doing the similar exercise for individuals 2, 3, 4, we obtain:

$$b_{\alpha}(g,\phi) - b_{\alpha}^{[-2]}(g,\phi) = 1.244$$

$$b_{\alpha}(g,\phi) - b_{\alpha}^{[-3]}(g,\phi) = 1.146$$

$$b_{\alpha}(g,\phi) - b_{\alpha}^{[-4]}(g,\phi) = 0.988$$

Check that the key player is delinquent 1. Formula:

$$d_{1^*}(g,\phi) = \frac{b_{\alpha,1}(g,\phi)\sum_{j=1}^{j=4} m_{j1}(g,\phi)}{m_{11}(g,\phi)}$$

$$\mathbf{M} = (\mathbf{I} - \phi\mathbf{G})^{-1} = \begin{pmatrix} 1.5317 & 0.65646 & 0.65646 & 0.45952 \\ 0.65646 & 1.3802 & 0.61101 & 0.19694 \\ 0.65646 & 0.61101 & 1.3802 & 0.19694 \\ 0.45952 & 0.19694 & 0.19694 & 1.1379 \end{pmatrix}$$

$$m_{11}(g,\phi) = 1.5317$$

and

$$
\begin{aligned}
\sum_{j=1}^{j=4} m_{j1}(g,\phi) &= m_{11}(g,\phi) + m_{21}(g,\phi) + m_{31}(g,\phi) + m_{41}(g,\phi) \\
&= 1.5317 + 0.65646 + 0.65646 + 0.45952 \\
&= 3.3041
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d_{1^*}(g,\phi) &= \frac{b_{\alpha,1}\sum_{j=1}^{j=3} m_{j1}(g,\phi)}{m_{11}(g,\phi)} \\
&= \frac{0.66521 \times 3.3041}{1.5317} \\
&= 1.435
\end{aligned}
$$

$$
d_{1^*}(g,\phi) = b_\alpha(g,\phi) - b_\alpha^{[-1]}(g,\phi) = 1.435
$$

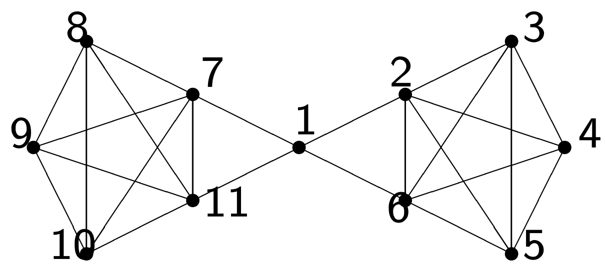
Is the key player always the more active criminal?

Holding $b_i(g, \phi)$ fixed, the intercentrality $d_i(g, \phi)$ of player i *decreases* with the proportion $m_{ii}(g, \phi)/b_i(g, \phi)$ of i's Bonacich centrality due to self-loops, and *increases* with the fraction of i's centrality amenable to out-walks.

Not always true.

Consider this network g with eleven criminals.

Figure 1: A bridge network



We distinguish three different types of equivalent actors in this network, which are the following:

Type	Criminals
1	1
2	2, 6, 7 and 11
3	3, 4, 5, 8, 9 and 10

Role of location in the network

Criminals are ex identical: $\boldsymbol{\alpha} = \mathbf{1}$

$\mathbf{b_1}(g, \phi) = (\mathbf{I} - \phi\mathbf{G})^{-1}\mathbf{1}$

$y_i^* = b_{1_i}(g, \phi)$ and $d_{i^*}(g, \phi) = b_1(g, \phi) - b_1^{[-i]}(g, \phi)$.

Take $\phi = 0.2$.

Table 1a: Key player versus Bonacich centrality in a bridge network

Player Type	1	2	3
$y_i = b_i$	8.33	9.17*	7.78
d_i	41.67*	40.33	32.67

Conclusion

We decompose cross effects into global and local externalities. We get a network.

We relate individual behavior to the network topology: Nash is Bonacich.

We identify geometrically the network Achille's heel, the *key player* is inter-centrality.

A number of extensions

With C^2 utilities, the Bonacich vector is a first-order approximation of interior equilibria for the Jacobian of $\nabla \mathbf{u}$.

When the planner's objective is welfare: $W^*(\mathbf{\Sigma}) \propto \sum_i x_i^*(\mathbf{\Sigma})^2$.

When the planner's objective is a group: group inter-centrality.

When the planner's tool is a tax of a subsidy to an optimal target.

Endogenous network with a two-stage game.

Joining delinquency networks

Equilibrium networks Allow individuals to choose whether they want to participate in the crime market or not in the first stage.

Consider the following two-stage game.

Fix an initial network g connecting agents.

In the first stage, each agent $i = 1, ..., n$ decides to enter the labor market or to become a delinquent.

Second stage, those who become criminals play the effort game.

Utility

$$u_i(\mathbf{x},g) = (1-\pi)\, x_i - \delta x_i^2 - \delta \sum_{j \neq i}^{n} x_i x_j + \pi\phi \sum_{j=1}^{n} g_{ij} x_i x_j$$

Nash equilibrium (before KP policy)

If $\theta\rho\,(g) < 1$, then there exists a unique Nash equilibrium $\mathbf{x}^*$:

$$\mathbf{x}^* = \frac{1-\pi}{\delta\,[1 + b(g,\theta)]}\mathbf{b}(g,\theta)$$

First stage:

$c_i \in \{0, 1\}$ denote i's decision, where $c_i = 1$ (resp. $c_i = 0$) stands for becoming a delinquent (resp. entering the labor market).

Agents entering the labor market earn a fixed wage (non-negative scalar) $\omega > 0$.

Definition 2 *The extended game is a two stage game where:*

- *In stage 1, each player $i \in N$ decides whether to participate ($c_i = 1$) or not ($c_i = 0$) to the crime market.*

- *In stage 2, let S be the set of players who decided to participate. Then, these players play the game in g_S.*

- *The final utilities are:*

$$U_i(S, \mathbf{x}_S, g) = \begin{cases} u_i(\mathbf{x}_S, g_S) & \textit{if} \quad i \in S \\ \omega & \textit{otherwise} \end{cases}$$

Definition 3 *The set S is supported in equilibrium if there exists a ω and a subgame perfect equilibrium where the set of players who decide to participate is S, given the outside option ω. S is also called an (equilibrium) participation pool of the game at the wage level ω.*

Let $\mathcal{E}(\omega)$ be the family of sets supported by ω at equilibrium in the extended game.

Proposition 5 *Let $S \subseteq N$ and $\theta\rho(g) < 1$ for all $j \in N \backslash S$. Then, the set S is supported at equilibrium by the outside option ω if and only if:*

$$\max_{j \in N \backslash S} \frac{b_j(g_{S \cup \{j\}}, \theta)}{1 + b(g_{S \cup \{j\}}, \theta)} \leq \frac{1}{1-\pi}\sqrt{\omega\delta} \leq \min_{i \in S} \frac{b_i(g_S, \theta)}{1 + b(g_S, \theta)}$$

Whenever an equilibrium exists, multiplicity of equilibria is a natural outcome of the extensive form game.

Finding the key player with criminal participation decision Given that the outside option ω is fixed, it is clear that the two-stage game is *supermodular,* in the sense that the payoffs of player i are increasing with respect to participation decisions of other agents.

Formally, for all $S \subseteq T \subseteq N$ and $i \in N \backslash T$, it is clear that:

$$b_i(g_{S \cup \{i\}}, \theta) \leq b_i(g_{T \cup \{i\}}, \theta)$$

Given that this game usually displays **multiple subgame perfect equilibria** in the endogenous delinquency network game, we define $x^*(g, \omega)$ to be the maximum aggregate equilibrium delinquency level when the delinquency network is g and the labor market wage is ω.

This delinquency level is equal to the total amount of delinquency in the worst case scenario of maximum delinquency.

Let i be an active delinquent, that is $c_i = 1$.

Suppose that delinquent i switches his current decision to $c_i = 0$, that is, delinquent i drops out from the delinquency pool and enters the labor market instead.

The binary decision profile then becomes $\mathbf{c} - \boldsymbol{\nu}^i$, and the new set of active delinquents is $C(\mathbf{c} - \boldsymbol{\nu}^i) = C(\mathbf{c}) \backslash \{i\}$.

The drop out of delinquent i from the delinquency pool also alters the network structure connecting active delinquents, as any existing direct link between i and any other delinquent in $C(\mathbf{c})$ is removed.

The new network connecting active delinquents is then $g(\mathbf{c})^{-i} = g(\mathbf{c} - \boldsymbol{\nu}^i)$, and the aggregate delinquency level becomes:

$$x^*(\mathbf{c} - \boldsymbol{\nu}^i) = \frac{1-\pi}{\delta} \frac{b\left(g(\mathbf{c} - \boldsymbol{\nu}^i), \theta\right)}{1 + b\left(g(\mathbf{c} - \boldsymbol{\nu}^i), \theta\right)}$$

The key player problem acquires a different shape in the setting with endogenous formation of delinquency pools.

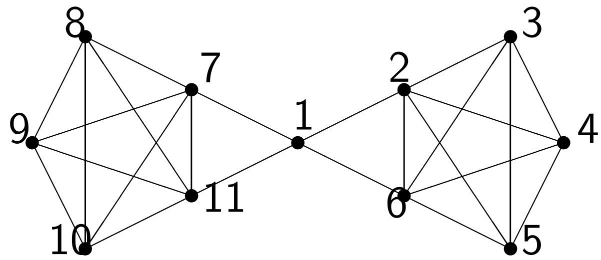
Initially, the planner must choose a player to remove from the network (first stage $t = 0$).

Then, players play the two-stage delinquency game.

First (second stage, $t = 1$), they decide whether to enter the delinquency pool or not.

Second (third stage, $t = 2$), delinquents choose how much effort to exert.

Consider again the network with eleven players delinquents.



Recall that, when $\theta = 0.2$ and the network of delinquents is exogenously fixed (or, equivalently, the outside option is $\omega = 0$), the key player was the player acting as a bridge, i.e. delinquent 1.

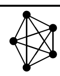
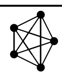
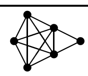
Consider the endogenous delinquency network formation in the two-stage game.

For low wages, player 1 is also the key player.

When ω becomes higher, delinquent 2 becomes the key player.

Table: KP for two different values of ω

Highest aggregate delinquency that results from eliminating this key player

	$\omega = 0.001$	$\omega = 0.003$
$x^*(g_{-1}, \omega)$	0.7843	0.7843
$x^*(g_{-2}, \omega)$	0.7847	0.7785
Key Player	1	2
Final delinquency pool	 	

When outside opportunities are high enough, all players from the same side of the player being removed do not have enough incentives to enter the delinquency pool at the first stage of the game.

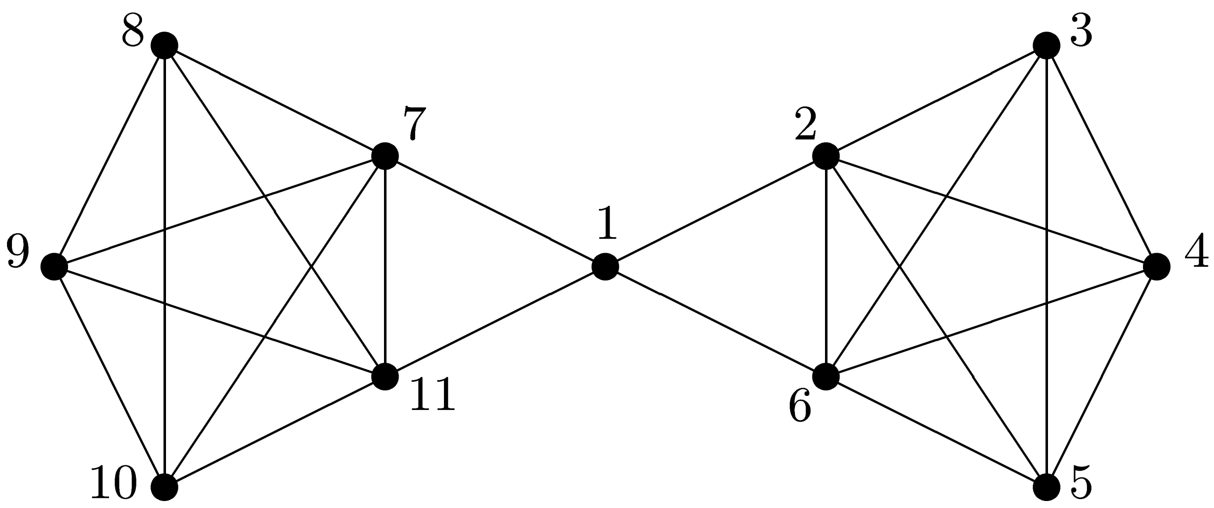
Hence, we do not get a "large" equilibrium with many players, and this constitutes an advantage for the planner who will choose to delete node 2.

How one policy (providing a higher ω) increases the effectiveness of another policy (choosing the key player) in order to reduce delinquency.

These policies are *complementary* from the point of view of their effects on total delinquency, although we are aware that they may be substitute if we had considered a budget-restricted planner who had to implement costly policies.

From individual key player to key group

The key group maximizes a group inter-centrality measure $c_S(\mathbf{g}, a)$.



S	θ_S
$\{2, 7\}^*$	67.22
$\{2, 8\}$	64.01
$\{3, 8\}$	59.39
$\{1, 2\}$	56.66
$\{2, 6\}$	50.41
$\{2, 3\}$	46.96
$\{3, 4\}$	42.15

Network Games under Strategic Complementarities

Mohamed Belhaj[1] Yann Bramoullé[2] Frédéric Deroïan[3]

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[1]Aix-Marseille School of Economics (École Centrale Marseille)
[2]Université Laval and Aix-Marseille School of Economics
[3]Aix-Marseille School of Economics

Introduction

- We analyze network games under strategic complementarities.
  - Agents are embedded in a fixed network and interact with their network neighbors.
  - They play a game characterized by linear best-replies and positive interactions.
  - An agent tends to increase his action if his neighbors increase theirs.
- Our main new assumption: Actions are bounded from above.
  - Natural in many contexts (time).
  - Actually hard to think about contexts where actions could be unbounded.

What did we know?

- With no upper bound, game analyzed in Ballester, Calvó-Armengol & Zenou (ECA 2006).
- They show the emergence of two domains.
  - Under small network effects, unique equilibrium where actions related to Bonacich centrality.
  - Under large network effects, no equilibrium exists due to explosive positive feedbacks.

What did we know?

- In their empirical implementation, Calvó-Armengol, Pattachini & Zenou (RES 2009) discuss the possibility of adding bounds.
    - "Let us bound the strategy space in such a game rather naturally by simply acknowledging the fact that students have a time constraint and allocate their time between leisure and school work. In that case, multiple equilibria will certainly emerge, which is a plausible outcome in the school setting" (p.1254)
- Reasonable conjecture under strategic complements and large network effects.

What do we find?

- We show that this conjecture does not hold.
  - A unique equilibrium always exists.
- We import powerful results from the theory of supermodular games.
  - Lattice structure of the equilibrium set.
  - Monotone comparative statics.
  - Fast algorithms to compute the equilibrium.

Is uniquenes surprising?

- General tendency of supermodular games to yield multiple equilibria.
  - E.g. Vives (JME 1990), Milgrom & Roberts (ECA 1990).
- Also, general tendency of network games to yield multiple equilibria.
  - Under strategic substitutes, or a mix of substitutes and complements, multiplicity is the norm, see Bramoullé, Kranton, D'amours (WP 2011).
- Somehow, linearity and complementarities discipline each other.

Structural properties of the equilibrium

- We analyze how an agent's action depends on his position in the network.
  - We find that action and Bonacich centrality are generally not aligned: An agent who is less central can play a higher action.
  - We identify a number of cases where this alignement is preserved.
    - Includes regular graphs, nested split graphs, the line and line-like graphs.
- We show that large network effects can break the interdependence between agents.
  - When an agent reaches the upper bound, he stops transmitting influence across the network.

The model

- Each agent i chooses an action x_i such that $0 \leq x_i \leq L$.
- The n agents are connected through a network $\mathbf{G}$.
  - The network may be weighted $g_{ij} \geq 0$ and directed $g_{ij} \neq g_{ji}$ and has no self-loop $g_{ii} = 0$.
- Agents play a game with best reply given by:

$$f_i(\mathbf{x}_{-i}) = \min(a_i + \delta \sum_j g_{ij} x_j, L)$$

- A Nash equilibrium $\mathbf{x}$ is a profile such that $\forall i, x_i = f_i(\mathbf{x}_{-i})$.

The model

- For instance, game Γ with quadratic payoffs

$$u_i(x_i, \mathbf{x}_{-i}) = -\frac{1}{2}x_i^2 + a_i x_i + \delta \sum_j g_{ij} x_i x_j$$

- More generally, any game with payoffs:

$$\pi_i(x_i, \mathbf{x}_{-i}) = v_i(x_i - a_i - \delta \sum_j g_{ij} x_j) + w_i(\mathbf{x}_{-i})$$

where v_i is increasing then decreasing and reaches its maximum at 0.

What do we know?

- When there is no bound $L = \infty$, two cases.
- If $\delta\lambda_{\max}(\mathbf{G}) < 1$, unique interior equilibrium where action is related to Bonacich centrality

$$
\begin{aligned}
\mathbf{x} &= (\mathbf{I} - \delta\mathbf{G})^{-1}\mathbf{1} \\
\mathbf{x} &= \mathbf{1} + \delta\mathbf{G}\mathbf{1} + \delta^2\mathbf{G}^2\mathbf{1} + \delta^3\mathbf{G}^3\mathbf{1} + \ldots \\
\mathbf{x} &= \mathbf{1} + \delta\mathbf{c}
\end{aligned}
$$

- If $\delta\lambda_{\max}(\mathbf{G}) \geq 1$, no equilibrium exists.
  - Because $\delta^t\mathbf{G}^t$ does not converge to zero.

Supermodularity

- When actions are bounded, the strategy space $[0, L]^n$ is a complete lattice.
- Because $\partial^2 u_i / \partial x_i x_j = \delta g_{ij} \geq 0$, the game with quadratic payoffs Γ is *supermodular*.
  - In particular, a Nash equilibrium always exists.
  - Moreover, Γ always has a smallest and a largest Nash equilibrium.
  - Properties hold for any other game with the same best-replies.

Uniqueness

Theorem. *When actions are bounded from above, there exists a unique Nash equilibrium.*

Uniqueness

- Sketch of the proof:
  - Consider the smallest equilibrium $\mathbf{x}^*$ and let $I = \{i : x_i^* < L\}$.
    - If $I = \varnothing$, the equilibrium is unique, so assume $I \neq \varnothing$.
  - Agents not in I play L in all equilibria. Fix their play at L and consider $\boldsymbol{\varphi}$ the restricted best-reply on $[0, L]^I$.
    - $\forall i \in I$, $x_i^* = a_i + \delta \sum_{j \in I} g_{ij} x_j^* + \delta \sum_{j \notin I} g_{ij} L$.
  - Introduce $b_i = a_i + \delta \sum_{j \notin I} g_{ij} L$. We have:

$$(\mathbf{I} - \delta \mathbf{G}_I)\mathbf{x}_I^* = \mathbf{b}.$$

  - Since this system has a positive solution, $\delta \lambda_{\max}(\mathbf{G}_I) < 1$.
  - This implies that $\boldsymbol{\varphi}$ is contracting. For any equilibrium $\mathbf{x}$, $\mathbf{x}_I = \boldsymbol{\varphi}(\mathbf{x}_I)$. So unique equilibrium. $\square$

Key to the proof:

Show that the best-reply function is contracting on a **critical subset** of the original strategy space;

namely, the set of actions lying between the smallest and largest equilibrium.

Since any equilibrium belongs to that set, this property of partial contraction is sufficient to guarantee uniqueness.

More generally, any supermodular game with such a partially contracting best-reply has a unique equilibrium.

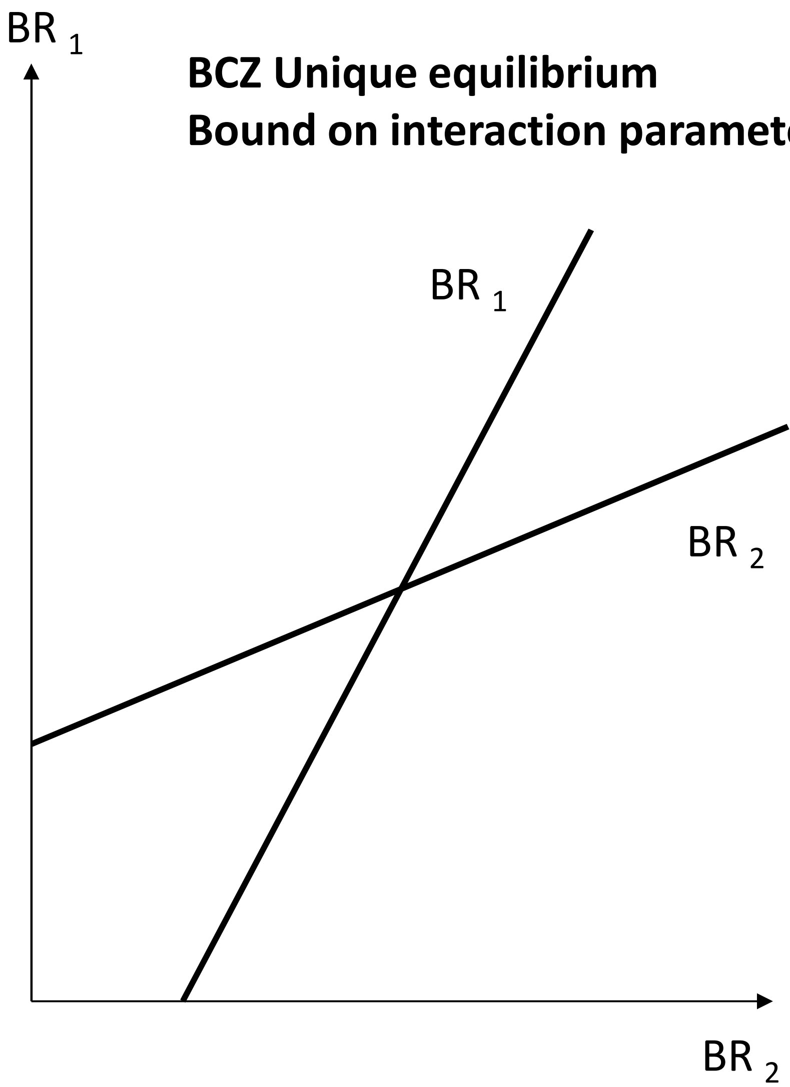
Therefore, uniqueness prevails even in the presence of large positive network effects.

The structure imposed by linearity somehow disciplines the natural tendency of strategic complementarities to generate multiple equilibria.

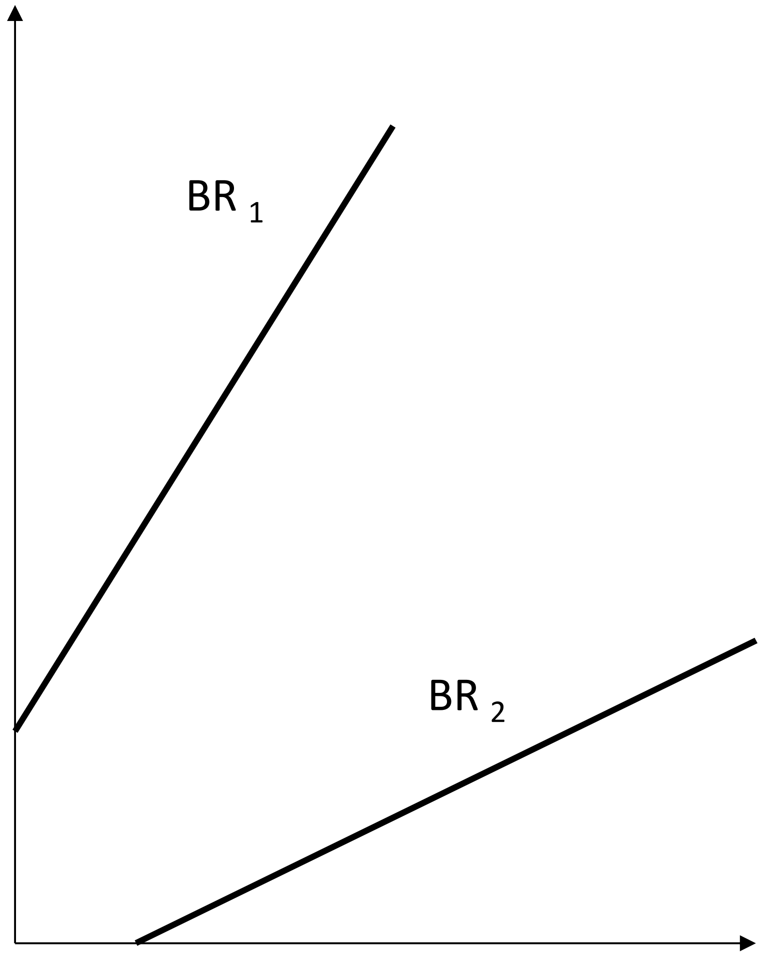
The structure imposed by the strategic complementarities somehow disciplines the tendency of linear network games to yield multiple equilibria.

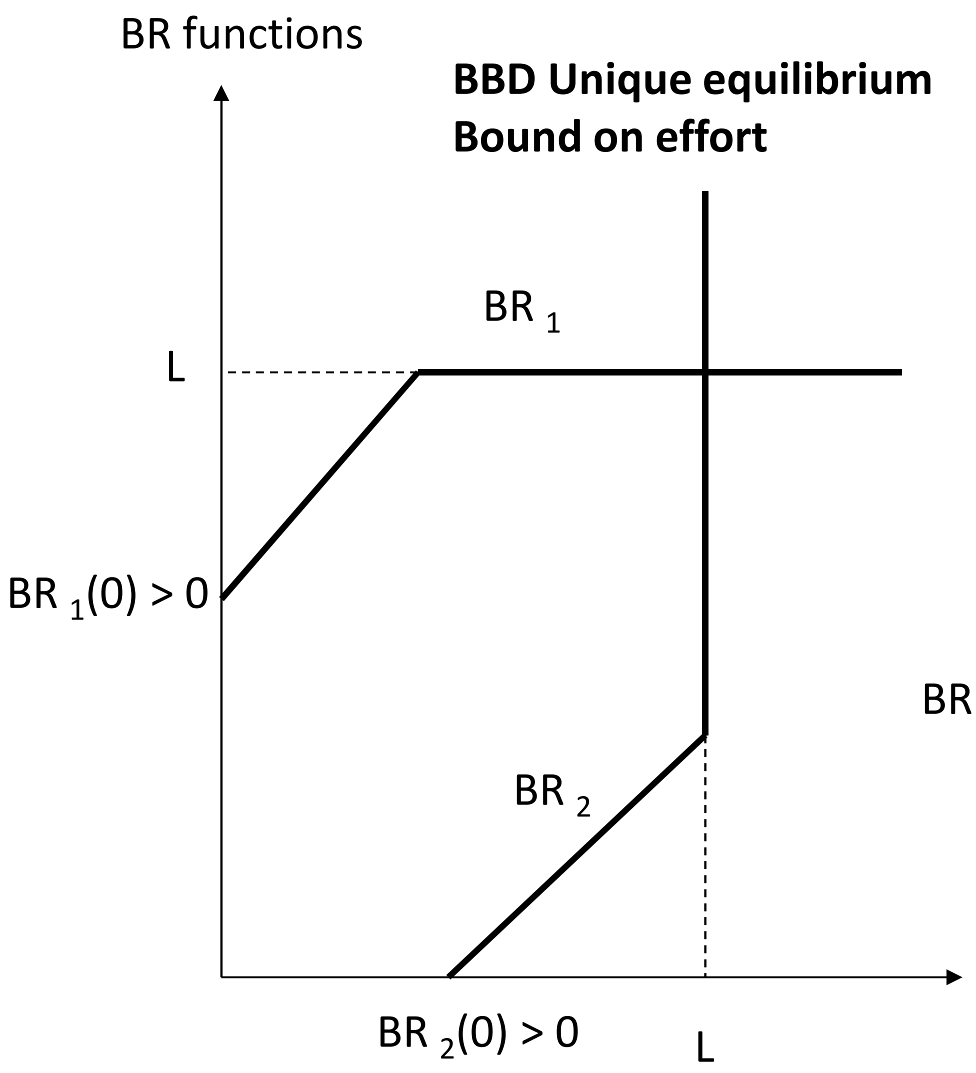
In short, linearity and complementarities discipline each other.

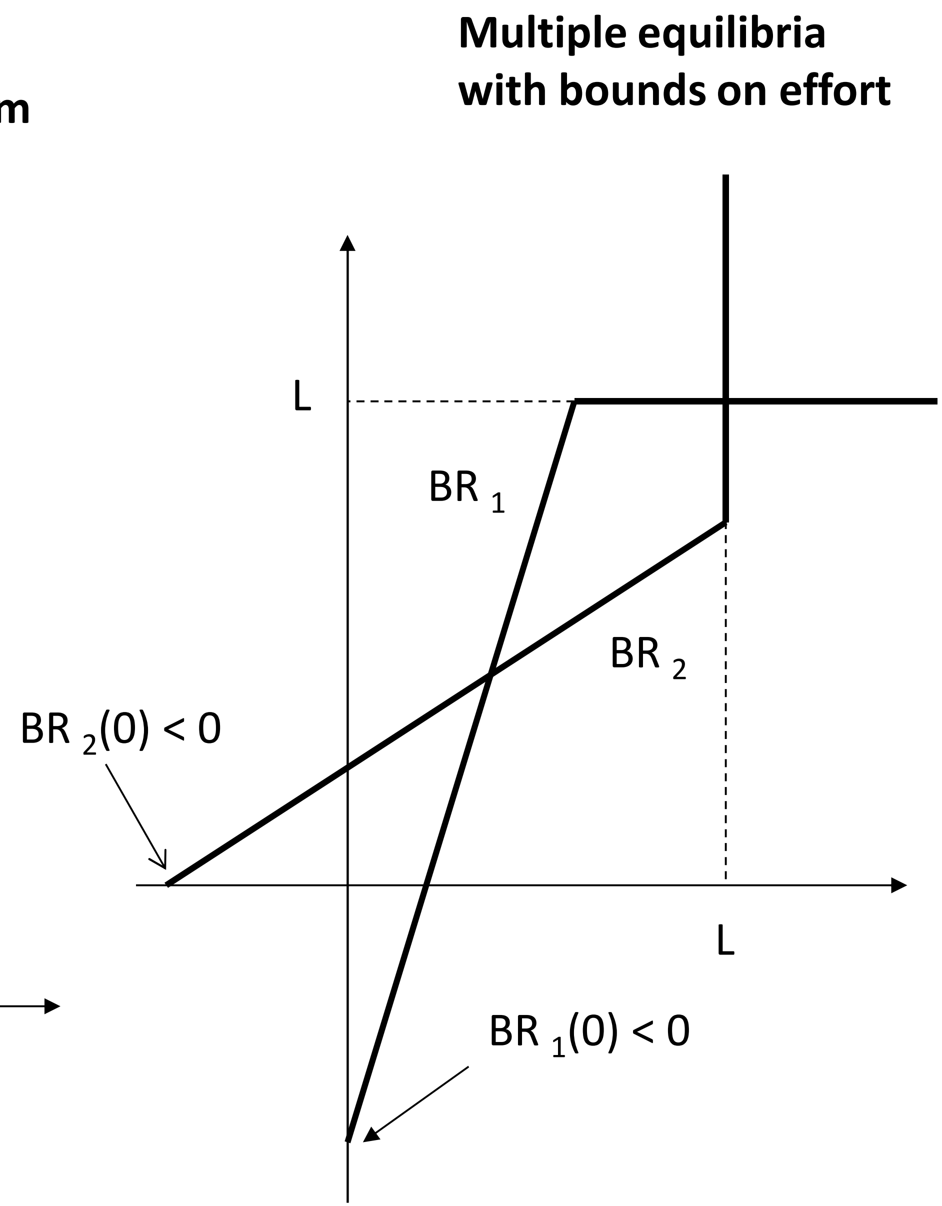
BCZ Unique equilibrium
Bound on interaction parameter



No upper bound No equilibrium




BR functions
BBD Unique equilibrium
Bound on effort
BR 1
L
BR 1(0) > 0
BR 2
BR 2(0) > 0
L


Multiple equilibria
with bounds on effort
L
BR 1
BR 2
BR 2(0) < 0
L
BR 1(0) < 0

Comparative statics

Corollary. *Individual action in the unique equilibrium* x_i^* *is weakly increasing in* δ, L, $\mathbf{a}$, *and* $\mathbf{G}$.

- Proof: By Theorem 6 in Milgrom & Roberts (1990). $\square$
- No need to use the implicit function theorem.
- Here, direct and indirect network effects are fully aligned.
  - If one agent increases his action, his network neighbors may only increase theirs.
  - In turn, their neighbors may only increase theirs and the effect propagates in the network.
- Very different from strategic substitutes, when direct and indirect affects are generally not aligned.
  - And comparative statics are much more complicated, see Bramoullé, Kranton & D'amours (WP 2011).

Comparative statics

- In particular, every non-isolated agent eventually reaches L as δ increases.
  - Once he reaches the upper bound, he stays there.

Corollary. *There are two threshold levels* δ_1^* *and* δ_2^* *such that some, but not all, agents play the upper bound iff* $\delta_1^* < \delta \leq \delta_2^*$

- We see three domains emerging.
  - $\delta_1^* = \inf\{\delta : \exists i, [(\mathbf{I} - \delta \mathbf{G})^{-1}\mathbf{1}]_i \geq L\}$
    $\delta_2^* = \max_i [(L - a_i)/(Lk_i)]$

Thus, we see three domains emerging as a function of δ.

When $\delta < \delta_1$, the equilibrium is interior and action is proportional to Bonacich centrality in the network.

When $\delta_1 \leq \delta < \delta_2$, some agents have reached the upper bound L but others have not.

When $\delta \geq \delta_2$, all agents have reached the upper bound L and action does not depend on the network position.

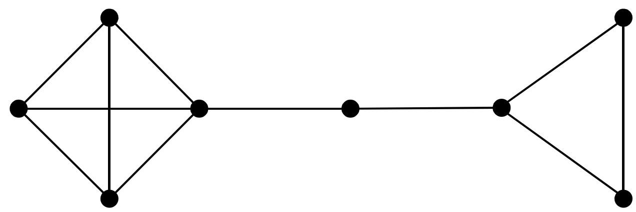
These two thresholds depend on the upper bound L and on the structure of the network.

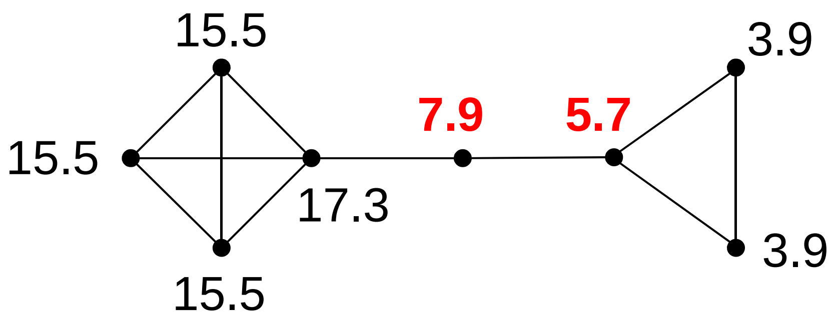
How to compute the equilibrium?

- From the literature on supermodular games, we can adapt fast algorithms to compute the equilibrium.
  - In particular, repeated myopic best-replies converge monotonically to the equilibrium starting from $\mathbf{x} = (0, 0, ...0)$ or $\mathbf{x} = (L, L, ..., L)$.
  - Faster if agents take turn in best-replying.

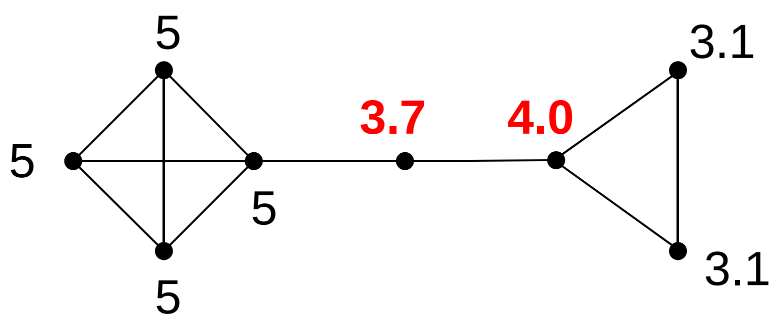
Network position and action

- We now study how network position is related to action.
  - Assume in what follows that $\forall i, a_i = 1$ and $g_{ij} \in \{0, 1\}$.
  - Agents only differ in their network characteristics.
  - Allows to clearly identify the effect of network position.
- We begin with an example showing that action and Bonacich centrality may not be aligned.
  - An agent who is more central may play a lower action.





δ=0.3   No bound



δ=0.3   L=5

When is action aligned with centrality?

- Say that i's neighborhood is nested in j's neighborhood if any neighbor of i is also a neighbor of j.

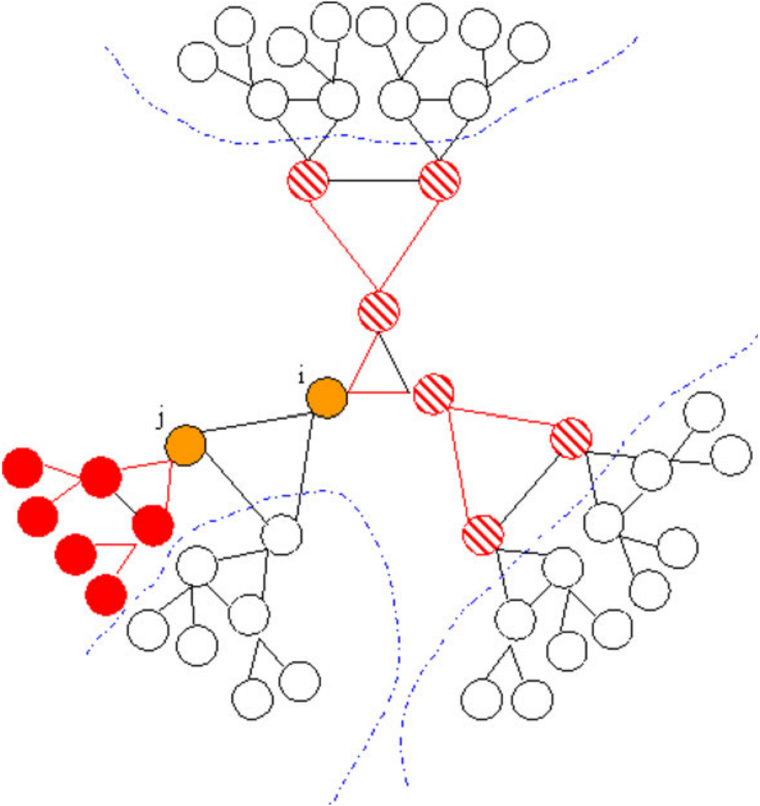
Proposition: *If i's neighborhood is nested in j's neighborhood, then* $x_i^* \leq x_j^*$

- In "nested split graphs", agents reach the upper bound in the order of their degrees.
  - Include stars.
  - Graphs that appear naturally in centrality-based network formation processes, see König, Tessone & Zenou (WP 2009).

When is action aligned with centrality?

- Action and centrality can be aligned even if neighborhoods are not nested.
- In particular, we can apply the analysis of Belhaj & Deroïan (IJGT 2010).
  - They study communication efforts under strategic complements and indirect network interactions.
    - Direct network interactions particular case.
  - They focus on the line and "line-like" graphs with a clear notion of geometric centrality.
  - They show that more central agents play a higher action in the lowest and highest equilibrium.

Corollary. *On the line and on "line-like" graphs, agents who are geometrically more central play a higher action.*



i
j

When is action aligned with centrality?

Proposition: *On regular graphs of degree k, every agent plays $x_i^* = 1/(1-\delta k)$ if $\delta < (L-1)/(kL)$ and $x_i^* = L$ otherwise.*

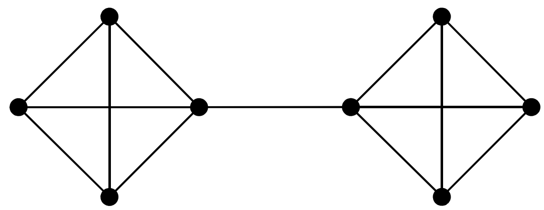
- All agents play the same action, even under large network effects.
  - The level at which the upper bound is reached is independent on the graph structure.

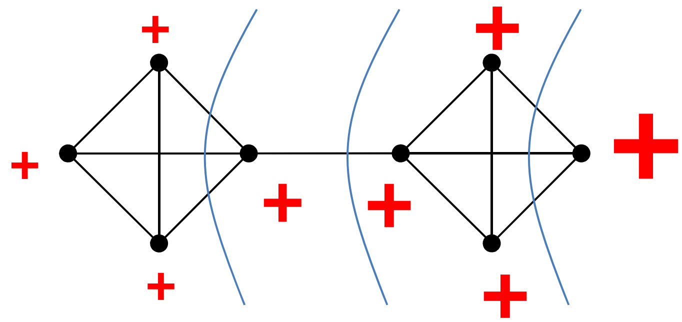
When is action aligned with centrality?

- Very different from strategic substitutes, see Bramoullé, Kranton, D'amours (WP 2011).
  - Agents play the same action only for small network effects.
  - Under large network effects, only asymmetric equilibria are stable even in fully symmetric graphs.
  - The level at which the symmetric equilibrium ceases to be stable depends on the graph's structure (through its lowest eigenvalue).

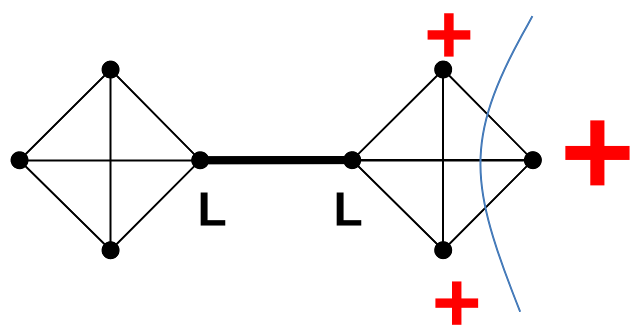
Broken interdependence

- We study the extent of interdependence under large network effects.
- Under small network effects, interdependence is "maximal". If $\mathbf{G}$ is connected,
$$\forall i, j, \frac{\partial x_i^*}{\partial a_j} > 0$$
- A shock on one agent affects the action of any other agent in society.
- We show that large network effects can break this interdependence.





$\delta < \delta_1$ *Full interdependence*



$\delta_1 \leq \delta < \delta_2$ *Broken interdependence*

Broken interdependence

- More generally, define $P_i = \{j : (\frac{\partial x_i^*}{\partial a_j})^+ > 0\}$ the set of agents who indirectly affect i.
  - A positive shock on j leads to an increase in i's action.

Proposition. *As δ increases, P_i shrinks monotonically towards $\varnothing$.*

- Idea of the proof:
  - $j \in P_i \Leftrightarrow$ there is a path of agents playing an interior action connecting i and j.

Broken interdependence

- Is interdependence broken quickly or not?
  - Depends on bridges and bridging agents.
  - In particular, if bridging agents also prominent within their communities, interdependence broken quickly.
  - But if bridging agents are relatively peripheral within, interdependence may last longer.

Conclusion: future research?

- Uniqueness means that an empirical implementation of the model should be relatively straightforward.

$$x_i = a_i + \delta \sum_j g_{ij} x_j + \varepsilon_i$$

  - With $\mathbf{G}$ and L known but δ to be estimated.
  - Multiplicity is one of the key difficulties in the econometrics of games.
  - Given assumptions on the error terms, in principle, we can compute the likelihood $L(x^* | \delta, G, L)$.

Conclusion: future research?

- Uniqueness may be, in some sense, non-generic.
  - We know that adding enough non-linearities or enough substituabilities leads to multiple equilibria.
  - Worse, the theory of supermodular games can only be applied when *all* strategic interactions are complements.
    - If for even one pair (i,j), we have $\partial f_i / \partial x_j < 0$, not one of the theorems holds.
- Is there any hope to develop a theory of "almost" supermodular games?
  - Are results robust to adding a little bit of non-linearities or substituabilities? Or not?

GAMES ON NETWORKS

Take the network as given and study the impact of network structure on outcomes.

GAMES WITH SUBSTITUABILITIES

Public Goods in Networks

Yann Bramoullé
Université Laval

Rachel Kranton
University of Maryland

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Introduction

- We study the first model of strategic experimentation in social networks.

- Individuals experiment to obtain new information *and* benefit from their neighbors' experimentation.

- Learning from others is a main source of learning in many situations:
    - Consumer choice.
    - R&D spillovers between firms.
    - Innovation adoption.

- Extensive literature in sociology and in marketing. Recent studies in economics.

- For instance, consider the adoption of a new crop in rural areas of developing countries.

- Conley & Udry (2003)

  - Look at the adoption of pineapple for export in Ghana.

  - Collected precise data on communication links. E.g. “Have you ever gone to X for advice about your farm?”.

  - “Our findings suggest that a farmer increases his fertilizer use after someone *with whom he shares information* achieves higher than expected profits when using more fertilizer than he did.”

  - Evidence of social learning. Does not take place at the village level. Example of information network.

- Foster & Rosenzweig (1995)

  - Look at the adoption of high-yield rice and wheat in India in the 1970’s.

  - “We find that farmers with experienced neighbors are significantly more profitable than those with inexperienced neighbors.”

- “farmers tend to free-ride on the learning of others” and “curtail their own costly experimentation” following an increase in the rate of adoption of their neighbors.

- Evidence of social learning. Yields strategic experimentation.

- We combine the two sets of findings.

- We study how the shape of the communication network affects experimentation patterns and welfare.

- We find strong network effects:

  - On the overall level of experimentation. May be lower in denser networks.

  - On individual experimentation. May be lower for individuals with a more central position in the network.

  - On the experimentation pattern. Networks lead to specialization and effort inequality.

  - On welfare. Inequal efforts may yield higher welfare when individuals who experiment are well-connected.

- New links increase access to information, but decrease incentives to experiment and may lower welfare.

- Contribution of the paper:

  - Introduces network aspects to the literature on strategic experimentation, Bolton & Harris (1999).

  - Endogenizes the generation of information in models of learning in networks, Bala & Goyal (1998).

  - Advances the economic theory of networks.

  - Studies the first model where a good is non-excludable among linked individuals.

  - Develops a new research strategy: Builds families of graphs to model different social structures.

Model

- Simple model.

- n individuals, Set of agents $N = \{1,\ldots,n\}$.

- Social network **g**, where $g_{ij} = 1$ indicates i and j are social neighbors.

- $N_i = \{j \in N - i : g_{ij} = 1\}$ Set of agents that are directly linked to agent i and $k_i = |N_i|$ number of i's neighbors

- Individuals can experiment to acquire information.

- Experimentation profile $\mathbf{e} = (e_1, \ldots, e_n)$. E.g. amount of land planted with a new crop.

- Individuals benefits from the experimentation results of their neighbors.

$$b\left(e_i + \sum_{j \in N_i} e_j\right)$$

where *b(.)* is increasing and concave.

- i.e., information diffuses one step, no decay.
- Constant marginal cost of experimentation c.
- Given **g**, individuals simultaneously choose their experimentation level e_i.

- Payoff for individual i:

$$U_i(\mathbf{e};\,\mathbf{g}) = b\left(e_i + \sum_{j \in N_i} e_j\right) - c_i e_i$$

Observe that:

$$\frac{\partial^2 U_i}{\partial {e_i}^2} = \frac{\partial^2 U_i}{\partial e_i \partial e_j} = b''\left(e_i + \sum_{j \in N_i} e_j\right) < 0$$

Thus e_i and e_j (local) strategic substitutes when $g_{ij} = 1$

- This defines a static game parametrized by the social network.

- How do the equilibria depend on the network?

Nash Equilibria

- Let e^* be such that $b'(e^*) = c$.

i.e.

$$e^* = b'^{-1}(c)$$

Experimentation for an isolated individual.

- Let $\bar{e}_i = \sum_{j \in N_i} e_j$ denote the information individual i receives from her neighbors, i.e. the total effort of i's neighbors.

Proposition: A profile ***e*** is a Nash equilibrium if and only if for every agent *i* either:

(1) $\bar{e}_i \geq e^*$ and $e_i = 0$
or
(2) $\bar{e}_i \leq e^*$ and $e_i = e^* - \bar{e}_i$

Proof. FOC: $$\frac{\partial U_i}{\partial e_i} = b'\left(e_i + \sum_{j \in N_i} e_j \right) - c = 0$$

This is equivalent to:

$$e_i + \sum_{j \in N_i} e_j = b'^{-1}(c) = e^*$$

$$\Leftrightarrow e_i = e^* - \sum_{j \in N_i} e_j$$

Therefore:

$$e_i = \begin{cases} 0 \quad if \quad e^* \leq \sum_{j \in N_i} e_j \\ e^* - \sum_{j \in N_i} e_j > 0 \quad otherwise \end{cases}$$

- Abstract characterization is easy. Geometric characterization is difficult.

- Experimentation levels are strategic substitutes.

- Distinguish three types of equilibria:

  A profile **e** is *specialized* when:
  Individuals exert $e_i = 0$ or $e_i = e^*$
  The agent $e_i = e^*$ is a *specialist*.

  A profile **e** is *distributed* when all individuals experiment, i.e. every agent exerts some positive effort, $0 < e_i < e^*, \forall i \in N$

  Hybrid equilibria fall between these two extremes

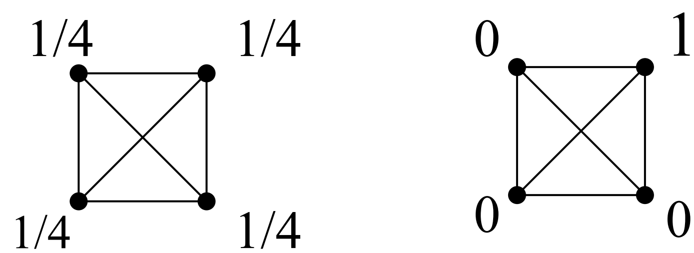
- Benefits for individuals who do not experiment may be greater than *b(e*)*.

- Indicates potential gains from specialization.

Illustration on Simple Graphs

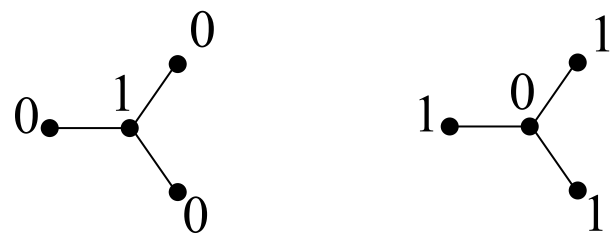
$e^* = 1$

- Complete graph (Nash equilibria)



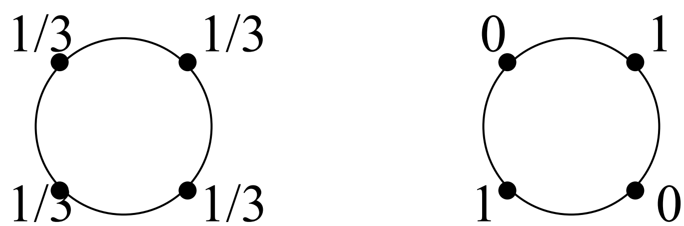
  - Symmetric, densely connected society.
  - Information is public.
  - Overall equilibrium experimentation (aggregate effort) is e^*, distributed in any way.

- Star



  - Asymmetric network.
  - All equilibria are specialized: (1) the center experiments, or (2) all agents in the periphery experiment.
  - In the right equilibrium, the center earns *b(3e*)*.

- Circle



  - Symmetric, not densely connected society.
  - Both distributed and specialized equilibria.
  - In the right equilibrium, individuals who do not search earn *b(2e*)*.

- What do we learn from these simple graphs?
  - The network is a main determinant of the equilibria.
  - The overall level of experimentation is usually indeterminate on incomplete networks.
  - Denser networks can lead to less overall experimentation.
  - Effort sharing is not always possible.
- In general networks, existence of equilibria guaranteed by standard arguments.
- However, this says nothing on their shape.

Definition: An independent set I of a network **g** is a set of players such that no two players who belong to I are linked, that is $\forall i, j \in I$ such that $i \neq j, g_{ij} = 0$. An independent set is maximal when it is not a proper subset of any other independent set.

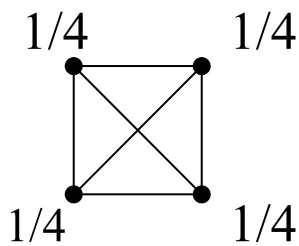
Any maximal independent set has the property that every player either belongs to it or is connected to a player that belongs to it.

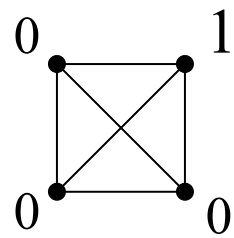
For any player i, there exists a maximal independent set I of the network **g** such that i belongs to I. This implies that any network **g** possesses at least one maximal independent set.

Given a graph **g**, we can define **a maximal independent set of order *r*** such that any individual not in I is connected to at least r individuals in I. That is, for a maximal independent set of order r, agents outside the set can have more than r, but no less than r, connections to agents in the set.

The case $r = 1$ simply corresponds to maximal independent sets. While every graph contains maximal independent sets, not all graphs contain maximal independent sets of higher order.

Consider first this Figure: complete graph with n=4

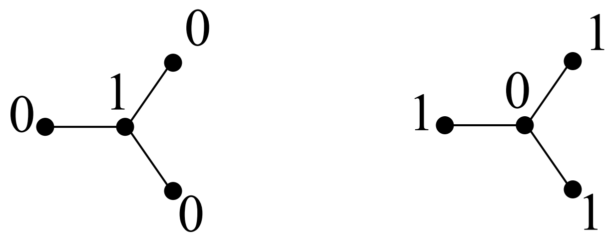




An independent set can include at most one player. There are thus four maximal independent sets, each including one player.

There is *no* maximal independent set of order *r* =2 or higher. This is a general property of complete graphs.

Consider now this Figure: a star network with n=3.

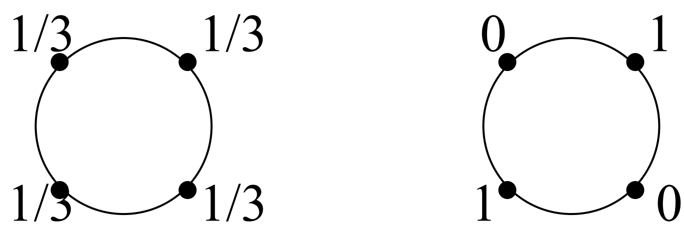


There are two maximal independent sets: the one that only includes the central player and the one that includes *both* peripheral players.
Observe that each peripheral player constitutes an independent set but it is *not* maximal.

There is however only one maximal independent set of order $r = 3$ composed by all peripheral players.

This is a general result of star-shaped graphs. If there are n players, then there is only one maximal independent set of order $r = n - 1$ composed of all peripheral players together.

Consider this Figure: a circle with n=4.



There are two maximal independent sets, each containing each player on opposite sides of the circle.

These two maximal independent sets are of order $r = 2$.

Go back to the model.

Because effort are strategic substitutes, maximal independent sets are a natural notion in this model. Indeed, in equilibrium, no two specialists can be linked. Hence, specialized equilibria are characterized by this structural property of a graph.

Specialists = *maximal independent set* of the graph.

Theorem 1: A specialized profile is a Nash equilibrium if and only if its set of specialists is a maximal independent set of the structure **g**. Since for every **g** there exists a maximal independent set, there always exists a specialized Nash equilibrium.

Proof:

Consider a specialized equilibrium where I is the set of specialists. Specialists play a best response if all their neighbors exert zero effort. This means that I is an independent set of the graph. A non specialist i plays a best response if

$$\sum_{j \in N_i} e_j \geq e^*$$

$$\Leftrightarrow \left| N_i \bigcap I \right| \geq 1$$

$$\Leftrightarrow \left| N_i \bigcap I \right| \geq r$$

This means that all players not in I are connected to at least r players in I. Combining both properties yields the result. Q.E.D.

Heuristic proof

- Take one agent i. Let her play e^*. Let all her neighbors play 0. Remove i and her neighbors.

- From remaining agents, take another agent j. Repeat the operation with j.

- Continue until all agents are covered.

- In the end, (1) no two specialists are linked, and (2) every non-specialist is connected to a specialist.

Q.E.D

- Specialized equilibria characterized by simple structural property of the graph.

Equilibrium selection: stable Nash equilibria

- Consider a simple notion of stability based on Nash tâtonnement.

See Fudenberg and Tirole (1991), *Game Theory*.

Definition: Define $f_i(\mathbf{e})$ as the best response of individual i to a profile $\mathbf{e} = (e_1,...,e_n)$ and define $\mathbf{f}$ as the collection of these individual best responses $\mathbf{f} = (f_1(\mathbf{e}),..., f_n(\mathbf{e}))$. Then, an equilibrium $\mathbf{e} = (e_1,...,e_n)$ is *stable* if and only if there exists a positive number $\rho > 0$ such that, for any vector $\boldsymbol{\varepsilon} = (\varepsilon_1,...,\varepsilon_n)$ satisfying $\forall i$, $|\varepsilon_i| \le \rho$ and $e_i + \varepsilon_i \ge 0$, the sequence $\mathbf{e}^{(n)}$, defined by $\mathbf{e}^{(0)} = \mathbf{e} + \boldsymbol{\varepsilon} = (e_1 + \varepsilon_1,..., e_n + \varepsilon_n)$ and $\mathbf{e}^{(n+1)} = \mathbf{f}(\mathbf{e}^{(n)})$, converges to $\mathbf{e} = (e_1,...,e_n)$.

This (standard) notion leads to a strong result:

- Only specialized equilibria are stable.

This result rests on the *strategic substitutability* of efforts of linked players.

Consider an equilibrium where everyone exerts some effort, and decrease the effort of an individual *i* by a small amount. Her neighbor(s) will adjust by increasing their own efforts. This increase can lead *i* to reduce his effort even more. In this case, the initial equilibrium is not stable.

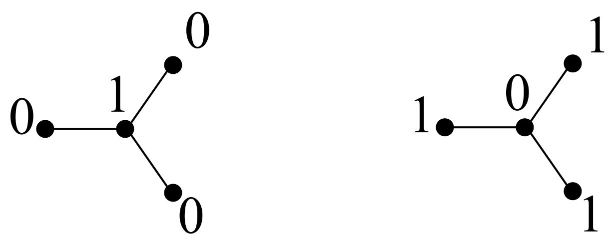
This process does not work in specialized equilibria when every agent *j* who exerts no effort is linked to two specialists. If we reduce the effort of these specialists, agent *j* will not adjust. He has access to two sources of information, and a small reduction will not lead him to increase his own effort.

Stable profiles thus correspond to maximal independent sets of order 2. Given a graph **g**, we
show a stable equilibria exists if and only if there is a maximal independent set of order 2.

Theorem 2: For any social structure **g**, an equilibrium is stable if and only if it is specialized and every non-specialist is connected to (at least) two specialists. Hence, there exists a stable equilibrium in a graph **g** if and only if it has a maximal independent set of order 2.

Thus an equilibrium is stable iff it is specialized and non-specialists are linked with at least two specialists.

Consider now the star network with n=3.



In both graphs, there is only one maximal independent set of order $r = 2$ composed by all peripheral players.

Consider the Nash equilibrium where the center exerts $e^* = 1$ and peripheral agents exert no effort (graph on the left).
The set of specialists is not a maximal independent set of order 2. Thus this equilibrium is *not* stable.

In contrast, consider the equilibrium where all peripheral agents exert effort (graph on the right). This equilibrium is stable, as the set of specialists is a maximal independent set of order 2.

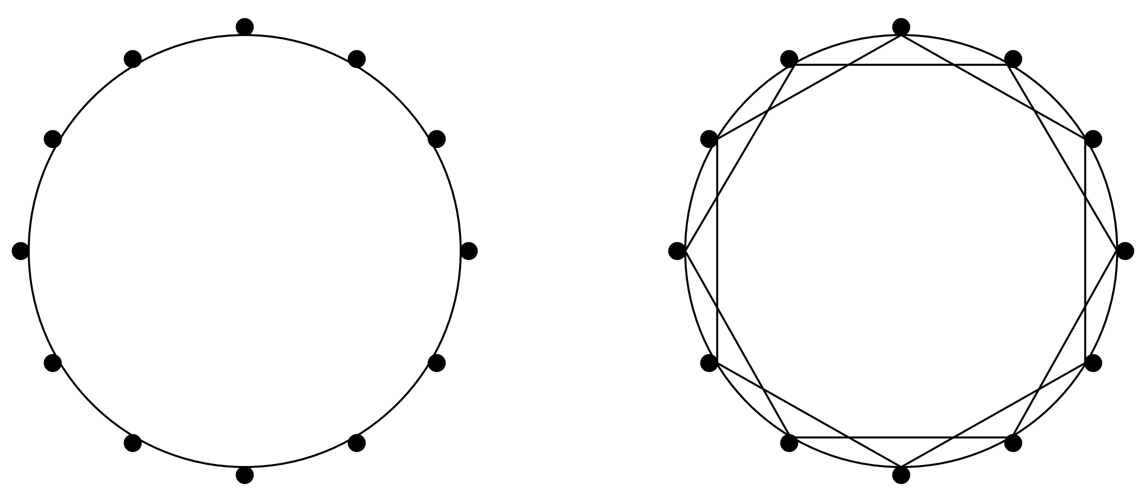
- E.g., on the star, experimentation by the center is not stable.
  → Better connected agents do less.

- Inequality in experimentation is a natural outcome of the network structure.

Models of Social Networks

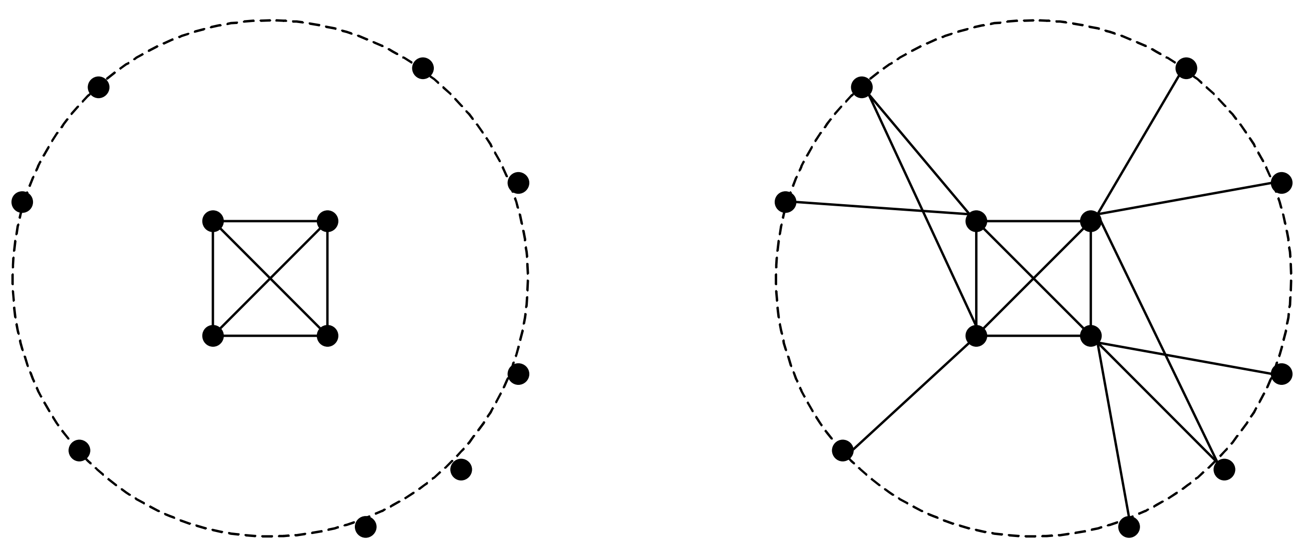
- In the paper, we build families of graphs representing different social structures.

- *Overlapping Neighborhoods*: symmetric structure where agents learn from those close in geographic or social space. Explore increasing levels of network density.



- *Communities/Bridges*: asymmetric structure with agents divided into disjoint communities. Explore increasing numbers of links – bridges – between communities.

- *Core-Periphery*: asymmetric, hierarchical structure where agents in periphery rely on core. Explore increasing density of links between core and periphery.



Welfare Analysis

- What are the welfare properties of the different equilibrium profiles?

- We adopt a simple utilitarian approach
$$W(\mathbf{e},\mathbf{g})=\Sigma_i\, U_i(\mathbf{e},\mathbf{g})$$

- Because of information externalities, no equilibrium yields first-best level of welfare.

- We study which Nash equilibria yield highest welfare.
  → *Second-best* profiles.

$$\frac{\partial W(\mathbf{e};\mathbf{g})}{\partial e_i} = b'\left(e_i + \sum_{j \in N_i} e_j\right) + \sum_{j \in N_i} b'\left(e_i + \sum_{j \in N_i} e_j\right) - c = 0$$

- Second term is *information premium* from specialization.

Compare with Nash equilibrium

$$\frac{\partial U_i}{\partial e_i} = b'\left(e_i + \sum_{j \in N_i} e_j\right) - c = 0$$

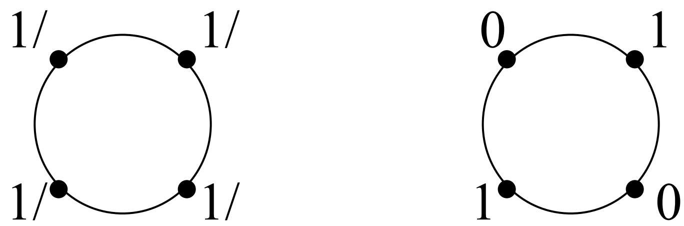
- A trade-off emerges :

* distributed equilibria have lower search costs.

* specialized equilibria can have information premia.

Overall experimentation is greater in specialized equilibria.

E.g. Circle



- Welfare of distributed equilibria

$$4\, b\, (e^*) - (4/3)\, ce^*$$

- Welfare of specialized equilibria

$$4\, b\, (e^*) + 2\, [b(2e^*) - b(e^*)] - 2\, ce^*$$

- Specialized equilibria are second-best when information premium exceeds additional search costs

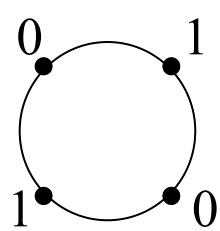
$$[b(2e^*) - b(e^*)] > (1/3)\, ce^*$$

Proposition: Specialized equilibria yield greater welfare if: (1) more information is sufficiently valuable, and (2) specialists are sufficiently well-connected.

$$\sum_{i\ specialist} k_i > n - 1$$

where k_i is the number of neighbors of i.

E.g. In circle, specialists together have 4 neighbors, and $n - 1 = 3$



- Similar method to compare specialized equilibria - count number of links between specialists and non-specialists.

What is the effect of adding a link?

- Consider a graph **G** and agents *i* and *j* not linked in **G**.

- We say a link between *i* and *j* leads to a welfare loss when second-best welfare level for **G** +*ij* is lower than that for **G**.

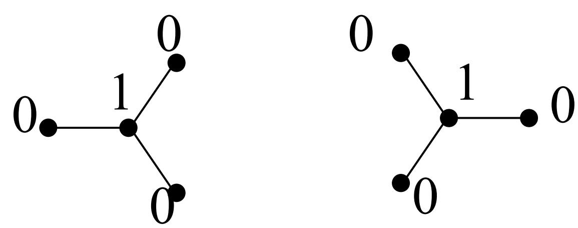
- Consider a second-best profile for **G:**

- *Benefit of New Link*: If *i* or *j* does not experiment, equilibrium is preserved. Link adds new source of information.

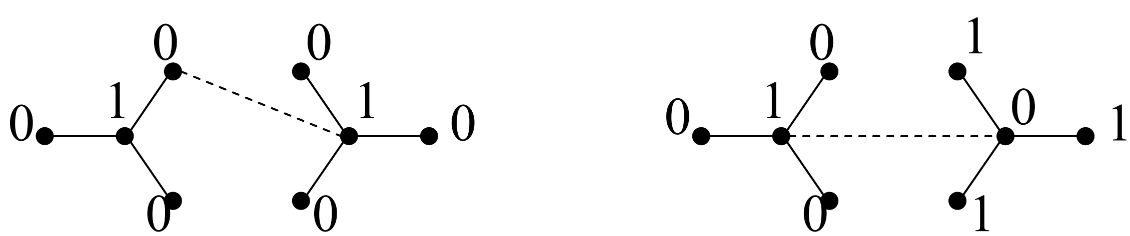
- *Cost of New Link*: If both *i* and *j* experiment, equilibrium is not preserved. Link is new source of information but leads to a loss of experimentation.

Proposition: A necessary condition for a loss in welfare is both agents experiment in all second-best profiles for **G**.

- Illustration: Two Stars



  - Experimentation by center individuals is unique second-best profile.



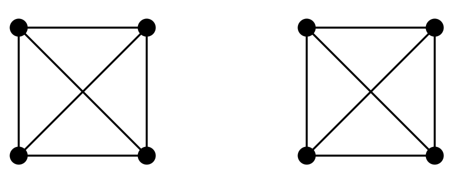
  - Linking center to periphery increases welfare.
  - Linking two centers can decrease welfare.

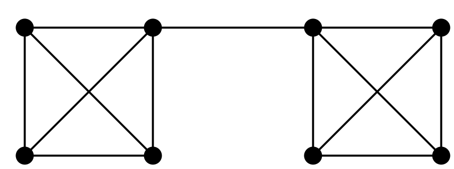
Bridges between Communities

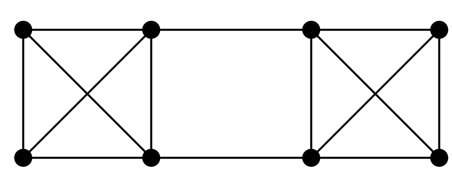
- Divide population into two communities, in which all agents are linked to each other.

- Some - but not all - agents are linked to agents in other community: *bridges* and *bridge agents*

- Members of this family of graphs described by β – the number of bridges.



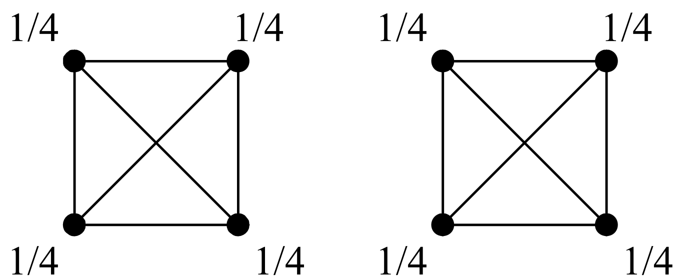


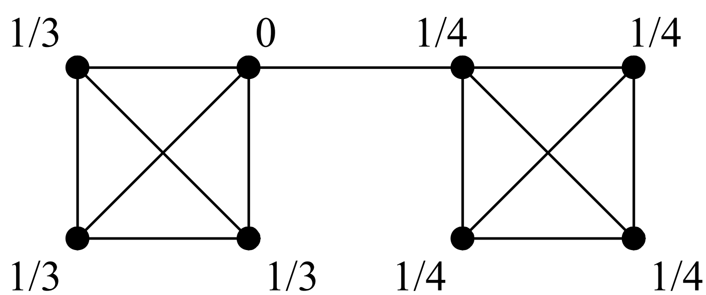


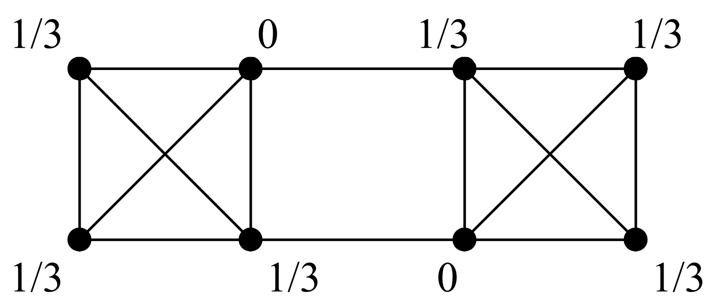
- Represents isolated villages, research units within firms....

Results:

- In equilibrium, in each community, overall experimentation is $e*$.
- For any two bridge agents, one does no experiment.
- Hence, across equilibria, average cost of non-bridge agents increases in β.







- Result counters conventional sociological wisdom about bridges – bridges help community by transmitting information.

- Here, bridge agents take advantage of other sources of information to reduce their effort.

- This reduction harms others in their community.

- This also illustrates our messages.

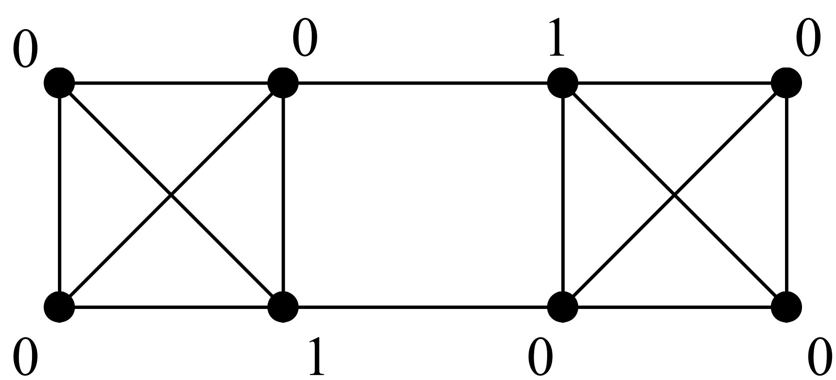
  → Better connected individuals experiment less, on average.

  → Equilibrium profiles become more inequal, on average, as β increases.

- Yet, on aggregate, welfare increases in the number of separate bridges – bridges that uniquely link agents across communities.

- Information premium for bridge agents who do not experiment.

- In second-best profiles, effort is concentrated on bridge agents.



→ Welfare is higher when experimentation is done by well-connected agents.

Negative Effect of New Non-Separate Bridge

- While separate bridges increase welfare, non-separate bridges can reduce welfare.
- Consider a structure where some agents already have links to the other community.
- In second-best profile, these agents experiment.
- Adding a link between them reduces their incentive to search and can lower welfare.